Mathematical Finance

Solution sheet 1

Solution 1.1

(a) Suppose that S is adapted and RCLL.

Regularity: Obviously, S_t^* is increasing in t. Fix $t_0 \geq 0$. Then for P-a.a. ω, $\lim_{t \downarrow t_0} S_t^*(\omega)$ exists in $[-\infty, \infty)$ and we have $\lim_{t \downarrow t_0} S_t^*(\omega) \geq S_{t_0}^*(\omega)$. Now by the right-continuity of $S(\omega)$, for every $\varepsilon > 0$ we can find $\delta > 0$ such that $|S_{t_0+\eta}(\omega) - S_{t_0}(\omega)| < \varepsilon$ whenever $0 \leq \eta \leq \delta$. Thus $|S_{t_0+\eta}^*(\omega) - S_{t_0}^*(\omega)| \leq \sup_{0 \leq \eta \leq \delta} |S_{t_0+\eta}(\omega) - S_{t_0}(\omega)| \leq \varepsilon$ whenever $0 \leq \eta \leq \delta$. This shows that $S^*(\omega)$ is right-continuous.

By a similar argument, it is easy to show that $S^*(\omega)$ has left limits.

Again fix $t_0 \geq 0$. It is easy to show that for P-a.a. ω, for each $T > 0$, $S(\omega)$ is bounded on $[0, T]$. From $|A_{t_0+\eta}(\omega) - A_{t_0}(\omega)| \leq \int_{t_0}^{t_0+\eta} |S_r(\omega)| \, \mathrm{d}r$, we even obtain that $A(\omega)$ is continuous.

Adaptedness: For each $t \geq 0$, since S is RCLL, we have $S_t^* = \sup_{r \in \mathbb{Q} \cap [0,t]} S_r$, showing that S_t^* is $\mathcal{F}_t$-measurable.

To show that A is adapted, consider $S^n = \sum_{k=1}^{\infty} \mathbb{1}_{\{(k-1)/n < t \leq k/n\}} S_{(k-1)/n}$. Because S is P-a.s. bounded on $[0, t]$, $S^n \to S$ uniformly on $[0, t]$ P-a.s. Now since $r \mapsto A_r$ is continuous, it suffices to show that for each $s < t$, A_s is $\mathcal{F}_s$-measurable. Set $K(n) = \sup\{k : k/n \leq s\}$. Then clearly

$$\int_0^s S_r \, \mathrm{d}r = \lim_{n \to \infty} \int_0^s S_r^n \, \mathrm{d}r = \lim_{n \to \infty} \sum_{k=1}^{K(n)} \frac{S_{(k-1)/n}}{n} \quad P\text{-a.s.}$$

So A_s is P-a.s. equal to an $\mathcal{F}_s$-measurable random variable and therefore $\mathcal{F}_s$-measurable because $\mathbb{F}$ is complete.

(b) If S is adapted and continuous, then by (a) both S^* and A are continuous and adapted. Therefore, $t \mapsto f(S_t, S_t^*, A_t)$ as a composition of continuous functions is continuous and adapted. Hence ϑ is predictable.

If S is only RCLL and adapted, the statement is not true. It is enough to consider an example where S is RCLL and adapted, but not predictable. An easy example on $[0, 1]$ is that $X_t = \mathbb{1}_{\{1/2 \leq t \leq 1\}} B$ with its natural filtration, where B is a (nondegenerate) Bernoulli random variable. Obviously X is RCLL and adapted (to its natural filtration). But since X is 0 on $[0, 1/2)$, any adapted and left-continuous process must be constant on $[0, 1/2]$. So the same should be true for any predictable process.

Now simply take $f(x, y, z) = x$, giving $\vartheta = S$, which is not predictable.

Solution 1.2 Start with a geometric Brownian motion with $\mu = 0$, $\sigma = 1$, so that $\bar{S}_t = \exp(W_t - t/2)$. It is clear that $\bar{S}_t \to 0$ as $t \to \infty$ a.s. So the stopping time $\bar{\tau} := \inf\{t \geq 0 : \bar{S}_t = 1/2\}$ is a.s. finite. Set $\psi(t) := \tan t$ and $S_t := \bar{S}_{\psi(t)}$ for $t \in [0, \pi/2)$ and $S_{\pi/2} = 0$. This yields again a continuous process S and a stopping time $\tau := \psi^{-1} \circ \bar{\tau}$ with $\tau \in [0, \pi/2)$ a.s. Thus we can use the predictable, self-financing strategy φ with $v_0 = 0$ and going short on $]\!]0, \tau]\!]$, i.e., $\vartheta_t := -\mathbb{1}_{]\!]0,\tau]\!]}(t)$ (which is adapted and left-continuous). It follows that

$$V(\varphi) = \int \vartheta \, \mathrm{d}S = -(S^\tau - S_0).$$

So we end up with $V_{\pi/2}(\varphi) = S_0 - S^\tau_{\pi/2} = S_0 - S_\tau = 1/2$, which gives an arbitrage.

Solution 1.3

(a) First note that the left-continuous function $\operatorname{sgn}(x)$ can be approximated pointwise by a sequence (g_n) of continuous functions and each $g_n(X)$ as a continuous adapted process is therefore predictable. Hence, as $\operatorname{sgn}(X) = \lim_{n\to\infty} g_n(X)$, we can conclude that $\operatorname{sgn}(X)$ is also predictable and of course bounded. In particular, this ensures that the stochastic integral $\int_0^\cdot \operatorname{sgn}(X)\, dX$ is well-defined. An alternative proof is to note that X as a continuous, adapted process is predictable and $\operatorname{sgn}(x)$ is Borel-measurable, so that the composition $\operatorname{sgn}(X)$ is also predictable.
Next, according to the given hint, we have a family of convex C^2-functions f_h such that $f_h(x) = -x$ for $x \leq 0$, $f_h(x) = x - h$ for $x \geq h$ and $f_h(x) \to |x|$, $f_h'(x) \to \operatorname{sgn}(x)$ for all x as $h \to 0$. Note that since f_h is convex, its first derivative f_h' is increasing and therefore by our construction it holds that $|f_h'(x)| \leq 1$ for all $x \in \mathbb{R}$.
Now applying Itô's formula for each $f_h(X)$, we obtain that

$$f_h(X_t) - f_h(0) - \int_0^t f_h'(X_s)\, dX_s = \frac{1}{2}\int_0^t f_h''(X_s)\, d[X]_s. \tag{1}$$

Since $\lim_{h\to 0} f_h'(x) = \operatorname{sgn}(x)$ and $|f_h'(x)| \leq 1$ for all $x \in \mathbb{R}$, the dominated convergence theorem for stochastic integrals implies that

$$\lim_{h\to 0}\int_0^\cdot f_h'(X_s)\, dX_s = \int_0^\cdot \operatorname{sgn}(X_s)\, dX_s$$

uniformly on any compact interval $[0, t]$ in probability (which will be denoted by u.c.p). Consequently, as $f_h(x) \to |x|$, we actually get

$$\lim_{h\to 0}\left(f_h(X_t) - f_h(0) - \int_0^t f_h'(X_s)\, dX_s\right) = |X_t| - |X_0| - \int_0^t \operatorname{sgn}(X_s)\, dX_s = L_t^X(0)$$

u.c.p. In view of (1), the equation above is equivalent to

$$L_t^X(0) = \lim_{h\to 0} \frac{1}{2}\int_0^t f_h''(X_s)\,d[X]_s,$$

u.c.p. Now note that $d[X]_s$ is a nonnegative (random) measure and $f_h'' \geq 0$ due to the convexity of f_h. So we can conclude that $t \mapsto \frac{1}{2}\int_0^t f_h''(X_s)\,d[X]_s$ is a increasing process for each $h > 0$ and therefore as the limit of increasing processes, $L_t^X(0)$ is also increasing in t.
Furthermore, by definition, we easily see that $L^X(0)$ has continuous trajectories and satisfies $L_0^X(0) = 0$. The latter together with the increasing property ensures that $L^X(0)$ is nonnegative.

(b) Clearly, $(X_t - K)^+ = \frac{1}{2}(|X_t - K| + (X_t - K))$. By the definition of the local time at K, we get immediately that

$$|X_t - K| = |X_0 - K| + \int_0^t \operatorname{sgn}(X_s - K)\,dX_s + L_t^X(K).$$

On the other hand, by Itô's formula, we have

$$X_t - K = X_0 - K + \int_0^t 1\,dX_s.$$

Hence, we obtain

$$\begin{aligned}(X_t - K)^+ &= \frac{1}{2}\Big(|X_0 - K| + (X_0 - K) + \int_0^t (1 + \operatorname{sgn}(X_s - K)\Big)\,dX_s + L_t^X(K)\Big)\\ &= (X_0 - K)^+ + \int_0^t \mathbb{1}_{\{X_s > K\}}\,dX_s + \frac{1}{2}L_t^X(K).\end{aligned}$$

Solution 1.4

(a) Apply Itô's formula to $S_t = s_0 \exp\left(\sigma W_t + (\mu - \frac{1}{2}\sigma^2)t\right)$ to see that S satisfies the desired dynamics $dS_t = S_t(\mu dt + \sigma dW_t)$.

(b) Note that $\{S_t > K\} = \Big\{W_t > \frac{1}{\sigma}(\log(K/s_0) - (\mu - \frac{1}{2}\sigma^2)t)\Big\}$. Since under the measure P, the random variable W_t has a normal distribution, we get

$$P[S_t > K] = P\left[W_t > \frac{1}{\sigma}(\log(K/s_0) - (\mu - \sigma^2/2)t)\right] > 0.$$

Similarly we have $P[S_t < K] > 0$.

(c) Let Q be an equivalent measure on $\mathcal{F}_T$ for S such that S is a martingale with respect to Q. Recall that (see Exercise 1.3, (b))

$$(S_t - K)^+ = (S_0 - K)^+ + \int_0^t 1_{\{S_s > K\}}\,dS_s + \frac{1}{2}L_t^S(K),$$

and also note that with S being a Q-martingale, the stochastic integral $\int_0^\cdot 1_{\{S_s>K\}}\,dS_s$ is also a Q-martingale. Hence, taking the Q-expectation of both sides of the equation above, we get

$$E_Q[(S_t-K)^+]-E_Q[(S_0-K)^+]=\frac{1}{2}E_Q[L_t^S(K)].$$

Since Q is equivalent to P, we can derive from (b) that $Q[S_t>K]>0$ and $Q[S_t<K]>0$. Consequently, since the function $g(x):=(x-K)^+$ is strictly convex on any interval containing K, Jensen's inequality applied for $E_Q[g(S_t)]$ is strict and therefore

$$\frac{1}{2}E_Q[L_t^S(K)]=E_Q[g(S_t)]-E_Q[g(S_0)]>g(E_Q[S_t])-g(s_0)=g(s_0)-g(s_0)=0.$$

It follows that $Q[L_t^S(K)>0]>0$ and of course also $P[L_t^S(K)>0]>0$.

(d) We first observe that the portfolio value at time $t>0$ is given by

$$V_t=\varphi_t^0\,1+\vartheta_tS_t=-K1_{\{S_t>K\}}+1_{\{S_t>K\}}S_t=\max(0,S_t-K)=(S_t-K)^+.$$

By definition, (φ^0,ϑ) is self-financing if and only if for any $t>0$,

$$V_t=V_0+\int_0^t\vartheta_s\,dS_s. \tag{2}$$

Now by Exercise 1.3, (b) and noting that $V_0=(S_0-K)^+$, we have

$$V_t=(S_t-K)^+=(S_0-K)^++\int_0^t 1_{\{S_s>K\}}\,dS_s+\frac{1}{2}L_t^S(K). \tag{3}$$

Thus, we see from the comparison of (2) with (3) that (φ^0,ϑ) is self-financing if and only if for any $t>0$, $L_t^S(K)$ is equal to zero P-a.s. But we know from (c) that $L_t^S(K)\geq 0$ P-a.s. and $P[L_t^S(K)>0]>0$, and hence (φ^0,ϑ) is not self-financing.