Mathematical Finance

Solution sheet 2

Solution 2.1 (a) By definition, NA means $\mathcal{G}(\Theta_{\mathrm{adm}}) \cap L^0_+ = \{0\}$. So assuming the latter gives that for all $\vartheta \in \Theta_{\mathrm{adm}}$,

$$H = G_T(\vartheta) = G_T(\vartheta) - 0 \geq 0 \; P\text{-a.s.} \implies G_T(\vartheta) = 0 \; P\text{-a.s.},$$

by setting $Y = 0$. Conversely, if $H = G_T(\vartheta) - Y \geq 0$ for some $\vartheta \in \Theta_{\mathrm{adm}}$ and $Y \in L^0_+$, then $G_T(\vartheta) = H + Y \geq 0$ P-a.s.. So $G_T(\vartheta) \in \mathcal{G}_T(\Theta_{\mathrm{adm}}) \cap L^0_+$ and NA implies $G_T(\vartheta) = 0$ P-a.s. which shows $H = Y = 0$ P-a.s.. Therefore $\mathcal{C}^0_{\mathrm{adm}} \cap L^0_+ = \{0\}$.

(b) The above proof gives $H = -Y \geq 0$ and therefore does not involve the admissibility. So the same argument works.

(c) Part (a) gives "$\Longrightarrow$". To prove the converse, take $H \in \mathcal{C}^0_{\mathrm{adm}} \cap L^0_+$ and set $H^n := H \wedge n \in (\mathcal{C}^0_{\mathrm{adm}} \cap L^\infty) \cap L^0_+$. To see this, note that $H = G_T(\vartheta) - Y$ because $H \in \mathcal{C}^0_{\mathrm{adm}}$ and $H \wedge n = H - (H - n)^+ = G_T(\vartheta) - (Y + (H-n)^+) = G_T(\vartheta) - Y'$. By assumption, $H^n = 0$ for all n and $H^n \to H$ P-a.s., giving $H = 0$ as desired.

(d) Recall that in Exercise 1.2, we constructed a process $S_t = \exp(W_{\tan t} - \tan t/2)$ for $0 \leq t < \pi/2$ and $S_{\pi/2} = 0$. Obviously, S is a local martingale on $[0, \pi/2]$, but not a martingale. Now we just take $\vartheta_t = -\mathbb{1}_{(0,\pi/2]}(t)$. Then

$$G_{\pi/2}(\vartheta) = -(S_{\pi/2} - S_0) = 1,$$

which gives an elementary arbitrage opportunity. Note that $G(\vartheta) = -(S - S_0) = 1 - S$ is not uniformly bounded from below. If $G(\vartheta)$ is uniformly bounded from below, then S is bounded from above. But $S \geq 0$ gives that S is bounded and therefore S must be a martingale, which is a contradiction.

Solution 2.2 (a) First $G_T(\vartheta) \in \mathcal{C}^0_{\mathrm{adm}}$ gives $G_T(\vartheta) \wedge n \in \mathcal{C}^0_{\mathrm{adm}} \wedge L^\infty$ for each $n \in \mathbb{N}$. So, $E[G_T(\vartheta) \wedge n] \leq 0$, $\forall n$. Note that $G_T(\vartheta) \wedge n \uparrow G_T(\vartheta)$ and $G_T(\vartheta) \geq -a$, so by monotone convergence, $E[G_T(\vartheta)] \leq 0$.

(b) Adaptedness is given directly. Because S is locally bounded, it suffices to show that if S is bounded, then S satisfies the martingale property. Note that since S is bounded, any bounded predictable process $\vartheta \in \Theta_{\mathrm{adm}}$, as is then $-\vartheta$; so part (a) gives $E_Q[-G(\vartheta)] \leq 0$ for all bounded predictable ϑ. Fix $m < n$ and $A \in \mathcal{F}_m$. Consider $\bar{\vartheta} := \mathbb{1}_A \mathbb{1}_{\{m<t\leq n\}}$. Then

$$E_Q[G_T(\bar{\vartheta})] = E_Q[\mathbb{1}_A(S_n - S_m)] = 0.$$

Since this holds for all $A \in \mathcal{F}_m$, the process S satisfies the martingale property under Q.

Solution 2.3 (a) Define

$$k^* := \min\{k \in \{1, \dots, N\} : G_{\tau_k}(\vartheta) \in L^0_+ \setminus \{0\}\},$$

and set $\sigma_0 := \tau_{k^*-1}$ and $\sigma_1 := \tau_{k^*}$. Observe that k^* is deterministic. Moreover, set

$$h := \begin{cases} h^{k^*} & \text{if } P[G_{\tau_{k^*-1}}(\vartheta) = 0] = 1, \\ h^{k^*} 1_{\{G_{\tau_{k^*-1}}(\vartheta) < 0\}} & \text{if } P[G_{\tau_{k^*-1}}(\vartheta) = 0] < 1. \end{cases}$$

In words, we wait until $G_\cdot(\vartheta)$ first becomes genuinely positive and then use the single step of ϑ from the previous τ_{k-1} on the set where the previous gains were zero or genuinely negative. Note that $P[G_{\tau_{k^*-1}}(\vartheta) < 0] > 0$ in the second case by the definition of k^*. We claim that $\vartheta^* := h 1_{\rrbracket \sigma_0, \sigma_1 \rrbracket} \in \mathbf{b}\mathcal{E}$ is an arbitrage opportunity. Indeed, in the first case,

$$G_T(\vartheta^*) = G_{\tau_{k^*}}(\vartheta) - G_{\tau_{k^*-1}}(\vartheta) = G_{\tau_{k^*}}(\vartheta) \in L^0_+ \setminus \{0\},$$

and in the second case,

$$\begin{aligned} G_T(\vartheta^*) &= \Big(G_{\tau_{k^*}}(\vartheta) - G_{\tau_{k^*-1}}(\vartheta)\Big) 1_{\{G_{\tau_{k^*-1}}(\vartheta) < 0\}} \\ &\geq -G_{\tau_{k^*-1}}(\vartheta) 1_{\{G_{\tau_{k^*-1}}(\vartheta) < 0\}} \in L^0_+ \setminus \{0\}. \end{aligned}$$

(b) Let $a \geq 0$ be such that $G_\cdot(\vartheta) \geq -a$ P-a.s. By right-continuity of the paths of $G_\cdot(\vartheta)$, it suffices to show $G_t(\vartheta) \geq -c$ P-a.s. for any $t \in [0, T)$. Seeking a contradiction, assume there is $t \in [0, T)$ such that $P[G_t(\vartheta) < -c] > 0$. But then $\vartheta^* := \vartheta 1_{\{G_t(\vartheta) < -c\} \times (t, T]}$ is predictable, S-integrable (see hints) and satisfies

$$\begin{aligned} G_\cdot(\vartheta^*) &= (G_\cdot(\vartheta) - G_t(\vartheta)) 1_{\{G_t(\vartheta) < -c\} \times (t, T]} \geq -a + c, \\ G_T(\vartheta^*) &= (G_T(\vartheta) - G_t(\vartheta)) 1_{\{G_t(\vartheta) < -c\}} \geq (-c - G_t(\vartheta)) 1_{\{G_t(\vartheta) < -c\}}. \end{aligned}$$

But this shows both that ϑ^* is admissible and that S fails NA, in contradiction to the hypothesis.

Solution 2.4 (a) By ii), $(L+M)^2 - [L+M]$ and $(L-M)^2 - [L-M]$ are UI martingales. So $LM - [L, M] = \frac{1}{4}((L+M)^2 - [L+M] - ((L-M)^2 - [L-M]))$ is a UI martingale. Also, as a difference of two increasing processes, $[L, M]$ is a finite variation process. Suppose B is anothe process satisfying the characterizing properties. Then $B - [L, M] = (B - LM) + (LM - [L, M])$ is a UI martingale, null at 0, and of finite variation. Moreover, $\triangle(B - [L, M]) = \triangle L \triangle M - \triangle L \triangle M = 0$. Therefore, $B - [L, M]$ is a continuous martingale and of finite variation, null at 0 and hence B and $[L, M]$ are indistinguishable.

(b) Symmetry follows directly from the definition. Suppose $L, M, N \in \mathcal{H}^2_0$. Then

$$(L+M)N - [L+M, N] = \frac{1}{4}(L+M+N)^2 - (L+M-N)^2 - ([L+M+N] - [L+M-N])$$

which is a UI martingale. Also,

$$\triangle[L+M,N] = \triangle(L+M)\triangle N = \triangle L\triangle N + \triangle M\triangle N = (\triangle[L,N] + \triangle[M,N]).$$

So, by the uniqueness of $[\cdot,\cdot]$, we have the bilinearity.

(c) Let τ be a stopping time. Clearly $L^\tau \in \mathcal{H}_0^{2,c}$ and $M^\tau \in \mathcal{H}_0^{2,d}$. So

$$E[L_\tau M_\tau] = E[L^\tau_\infty M^\tau_\infty] = 0,$$

and therefore LM is a UI martingale. We also have $\triangle[L,M] = \triangle L\triangle M = 0$, where we used that L is continuous. Thus $[L,M]$ is continuous. Since LM and $LM-[L,M]$ are both UI martingales, $[L,M]$ is also a UI martingale. Then $[L,M]$ is a continuous martingale of finite variation and null at 0, so we must have $[L,M]\equiv 0$.

(d) By bilinearity and stopping, we only need to show that if $M \in \mathcal{H}_0^{2,d}$ then

$$[M] = \sum_{0<s\le\cdot} (\triangle M_s)^2.$$

Denote the process on the right hand side by R. Let τ be a stopping time. Then

$$E[M_\tau^2 - R_\tau] = E[(M^\tau_\infty)^2 - R^\tau_\infty] = 0$$

by fact i) showing that $M^2 - R$ is a UI martingale. Since R is clearly an increasing, adapted, RCLL process, we just need to check the jump property. Indeed,

$$\triangle R_t = \sum_{0\le s\le t}(\triangle M_s)^2 - \sum_{0\le s\le t-}(\triangle M_s)^2 = (\triangle M_t)^2.$$

So by the uniqueness of $[M]$, we have $[M]=R$.

(e) Obviously, $X_0 = 0$ P-a.s.. For each $t\ge 0$, we have

$$\begin{aligned}
E[X_t] &= \sum_{n=1}^\infty E\left[\sum_{k=1}^n Y_k \middle| N_t = n\right] P[N_t=n] - \mu\lambda t\\
&= \sum_{n=1}^\infty E\left[\sum_{k=1}^n Y_k\right] e^{-\lambda t}\frac{\lambda^n t^n}{n!} - \mu\lambda t\\
&= \sum_{n=1}^\infty \mu n e^{-\lambda t}\frac{\lambda^n t^n}{n!} - \mu\lambda t\\
&= \mu\lambda t\sum_{n=0}^\infty e^{-\lambda t}\frac{\lambda^n t^n}{n!} - \mu\lambda t = 0,\\
E[X_t^2] &= \mathrm{Var}\left(\sum_{k=1}^{N_t} Y_k\right) = \mathrm{Var}(Y_k)E[N_t] + E[Y_k]^2\mathrm{Var}(N_t) = (\sigma^2+\mu^2)\lambda t.
\end{aligned}$$

So X_t for each $t\ge 0$ is square-integrable. Moreover, X is a martingale because of the independent and stationary increments and $E[X_t]=0$. Now note that since $Y_k \ne 0$ P-a.s., X has a jump iff N has a jump. Define $\tau_n := \inf\{t\ge 0 : N_t = n\}$.

We know that $\tau_n \sim \text{Gamma}(n, \lambda)$ and so $E[\tau_n] = n/\lambda$ and $\text{Var}(\tau_n) = n/\lambda^2$. Observe that $(X^{\tau_n})^2$ is submartingale. So

$$\begin{aligned}
\sup_{t\geq 0} E[(X_t^{\tau_n})^2] &\leq E[(X_\infty^{\tau_n})^2] \\
&= E[X_{\tau_n}^2] \\
&= E\left[\left(\sum_{k=1}^n Y_k - \mu\lambda\tau_n\right)^2\right] \\
&= \text{Var}\left(\sum_{k=1}^n Y_k - \mu\lambda\tau_n\right) + E\left[\sum_{k=1}^n Y_k - \mu\lambda\tau_n\right]^2 \\
&= n\sigma^2 + \frac{n\mu^2\lambda^2}{\lambda^2} + (n\mu - n\mu)^2 = n(\sigma^2 + \mu^2).
\end{aligned}$$

To compute $E[\sum_{s>0}(\triangle X_s^{\tau_n})^2]$, we observe that $\triangle X_{\tau_k}, 1 \leq k \leq n$ are i.i.d. and $\triangle X_{\tau_k} \overset{d}{=} Y_k$. So,

$$E\left[\sum_{s>0}(\triangle X_s^{\tau_n})^2\right] = E\left[\sum_{k=1}^n(\triangle X_{\tau_k})^2\right] = E\left[\sum_{k=1}^n Y_k^2\right] = n(\mu^2 + \sigma^2) = E[(X_\infty^{\tau_n})^2].$$

Therefore, we have $X^{\tau_n} \in \mathcal{H}_0^{2,d}$ and $X \in \mathcal{H}_{\text{loc}}^{2,d}$. The last claim follows from the proof of (d).