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Mathematical Finance

Solution sheet 3

Solution 3.1

(a) Recall that a simple Poisson process has a.s. increasing trajectories, so it is of finite variation. Therefore, a simple Poisson process is a good integrator.

Now we argue that Brownian motion is also a good integrator. Let $H^n \in b\mathcal{E}$ and $H^n \to 0$ uniformly in (ω, t). We claim that $I_B(H^n) \to 0$ in L^2. To this end, we compute

$$E[(I_B(H^n))^2] = E\left[\sum_{i=0}^{k(n)}\sum_{j=0}^{k(n)} h_i^n h_j^n (B_{\tau_{i+1}^n} - B_{\tau_i^n})(B_{\tau_{j+1}^n} - B_{\tau_j^n})\right].$$

For $i \neq j$, using the optional stopping theorem, we get (assuming $i < j$)

$$E[h_i^n h_j^n (B_{\tau_{i+1}^n} - B_{\tau_i^n})(B_{\tau_{j+1}^n} - B_{\tau_j^n})] = E[h_i^n h_j^n (B_{\tau_{i+1}^n} - B_{\tau_i^n}) E[(B_{\tau_{j+1}^n} - B_{\tau_j^n})|\mathcal{F}_{\tau_j^n}]] = 0.$$

So all cross terms vanish and we have

$$\begin{aligned}
E[(I_B(H^n))^2] &= E\left[\sum_{i=0}^{k(n)} (h_i^n)^2 (B_{\tau_{i+1}^n} - B_{\tau_i^n})^2\right] \\
&\leq \|H^n\|_\infty^2 E\left[\sum_{i=0}^{k(n)} (B_{\tau_{i+1}^n} - B_{\tau_i^n})^2\right] \\
&= \|H^n\|_\infty^2 E\left[\sum_{i=0}^{k(n)} B_{\tau_{i+1}^n}^2 - B_{\tau_i^n}^2\right] \\
&= \|H^n\|_\infty^2 E[B_T^2] \to 0.
\end{aligned}$$

(b) Suppose f has a jump at $x_0 > 0$. In particular, we may assume that f is right-continuous at x_0. By the continuity of f, we may assume that for some $\varepsilon > 0$, $f(x) < (f(x_0) + f(x_0-))/2$ for all $x \in [x_0 - \varepsilon, x_0)$. Define iteratively the stopping times $\tau_0 := 0$ and

$$\sigma_1 := \inf\{t \geq 0 : B_t = x_0\}, \tau_1 := \inf\{t \geq \sigma_1 : B_t \leq x_0 - \varepsilon\},$$
$$\sigma_k := \inf\{t \geq \tau_{k-1} : B_t = x_0\}, \tau_k := \inf\{t \geq \sigma_k : B_t \leq x_0 - \varepsilon\}.$$

For each $n \in \mathbb{N}$, we consider $H^n := \frac{1}{n}\sum_{i=1}^n \mathbb{1}_{]\!]\sigma_i,\tau_i]\!]}$. Clearly $H^n \in b\mathcal{E}$ and $\|H^n(\omega, t)\|_\infty = \frac{1}{n} \to 0$. But

$$I_X(H^n) = \frac{1}{n}\sum_{i=1}^n f(B_{\tau_i}) - f(B_{\sigma_i}) = \frac{1}{n}\sum_{i=1}^n f(x_0 - \varepsilon) - f(x_0) \leq -\frac{\triangle f(x_0)}{2} \not\to 0.$$

Therefore X is not a good integrator.

Solution 3.2

(a) Let $X \geq 0$ and $(\tau_n)_{n\in\mathbb{N}}$ a localizing sequence such that $\mathbb{1}_{\{\tau_n>0\}}X^{\tau_n}$ is a supermartingale. Fix $s < t$, $K > 0$, and let $A \in \mathcal{F}_s$. By the supermartingale property, we have

$$E[\mathbb{1}_{A\cap\{X_s^{\tau_n}\leq K\}}\mathbb{1}_{\{\tau_n>0\}}X_s^{\tau_n}] \geq E[\mathbb{1}_{A\cap\{X_s^{\tau_n}\leq K\}}\mathbb{1}_{\{\tau_n>0\}}X_t^{\tau_n}].$$

Applying the dominated convergence theorem on the LHS and Fatou's lemma on the RHS gives

$$E[\mathbb{1}_{A\cap\{X_s\leq K\}}X_s] \geq \liminf_{n\to\infty} E[\mathbb{1}_{A\cap\{X_s^{\tau_n}\leq K\}}\mathbb{1}_{\{\tau_n>0\}}X_t^{\tau_n}] \geq E[\mathbb{1}_{A\cap\{X_s\leq K\}}X_t].$$

Sending $K \to \infty$ and using the monotone convergence theorem give the supermartingale property. Now $E[X_t] \leq E[X_0] < \infty$ shows the integrability. The adaptedness is clear.

(b) First suppose that X is a nonnegative submartingale. Then for $t > 0$, for each stopping time $\tau \leq t$, we have $X_\tau \leq E[X_t|\mathcal{F}_\tau]$. This implies that the family $\{X_\tau : \tau \leq t \text{ stopping time}\}$ is UI.

Conversely, suppose that $X \geq 0$ is locally a submartingale and is of class DL. Let $(\tau_n)_{n\in\mathbb{N}}$ be a localizing sequence such that $\mathbb{1}_{\{\tau_n>0\}}X^{\tau_n}$ is a submartingale. Fix $s < t$. We have

$$\mathbb{1}_{\{\tau_n>0\}}X_{s\wedge\tau_n} \leq E[\mathbb{1}_{\{\tau_n>0\}}X_{t\wedge\tau_n}|\mathcal{F}_s].$$

Before we proceed, we prove the following

Lemma. Suppose $Y_n \to Y$ P-a.s. and $E[Y_n|\mathcal{F}] \leq E[Z|\mathcal{F}]$ for all $n \in \mathbb{N}$. If $(Y_n)_{n\in\mathbb{N}}$ is UI, then $E[Y|\mathcal{F}] \leq E[Z|\mathcal{F}]$.

Proof of Lemma. For all $B \in \mathcal{F}$, we have $E[\mathbb{1}_A Y_n] \leq E[\mathbb{1}_B Z]$ and $Y_n\mathbb{1}_B \to Y\mathbb{1}_B$ P-a.s.. Moreover, $(Y_n\mathbb{1}_B)_{n\in\mathbb{N}}$ is UI like $(Y_n)_{n\in\mathbb{N}}$, so that we get $E[Y\mathbb{1}_B] = \lim_{n\to\infty} E[Y_n\mathbb{1}_B] \leq E[Z\mathbb{1}_B]$. The result follows.

Now we get back to the main assertion. Since X is of class DL, we know that the family $\{X_{\tau_n\wedge t} : n \in \mathbb{N}\}$ is UI which also implies the uniform integratbility of the family $\{\mathbb{1}_{\{\tau_n>0\}}(X_{\tau_n\wedge t} - X_{\tau_n\wedge s}) : n \in \mathbb{N}\}$. Now applying the lemma, we get $E[X_t - X_s|\mathcal{F}_s] \geq 0$. Therefore, we establish the submartingale property. The class DL property also gives the integrability. The adaptedness is clear.

(c) Suppose X is a supermartingale. By definition, X_0 is integrable. Also $X^- = (-X)^+$ shows that X^- is a nonnegative submartingale. By part (b), X^- is of class DL.

Conversely, suppose that X_0 is integrable and X^- is of class DL. Let τ_n a localizing sequence such that $\mathbb{1}_{\{\tau_n>0\}}X^{\tau_n}$ is a supermartingale. We can rewrite the supermartingale property as

$$E[\mathbb{1}_{\{\tau_n>0\}}(X^+_{\tau_n\wedge t} - X^-_{\tau_n\wedge t})|\mathcal{F}_s] \leq \mathbb{1}_{\{\tau_n>0\}}X_{\tau_n\wedge s}.$$

which gives

$$E[\mathbb{1}_{\{\tau_n>0\}}X^+_{\tau_n\wedge t}|\mathcal{F}_s] \leq E[\mathbb{1}_{\{\tau_n>0\}}X^-_{\tau_n\wedge t}|\mathcal{F}_s] + \mathbb{1}_{\{\tau_n>0\}}X_{\tau_n\wedge s}.$$

Applying Fatou's lemma on the LHS yields

$$E[X^+_t|\mathcal{F}_s] \leq \liminf_{n\to\infty} E[\mathbb{1}_{\{\tau_n>0\}}X^+_{\tau_n\wedge t}|\mathcal{F}_s] \quad \leq \liminf_{n\to\infty} E[\mathbb{1}_{\{\tau_n>0\}}X^-_{\tau_n\wedge t}|\mathcal{F}_s] + \lim_{n\to\infty} \mathbb{1}_{\{\tau_n>0\}}X_{\tau_n\wedge s}.$$

Because X^- is of class DL, $\mathbb{1}_{\{\tau_n>0\}}X^-_{\tau_n\wedge t} \to X^-_t$ in L^1. By the P-a.s. uniqueness of the L^1-limit, we have $\liminf_{n\to\infty} E[\mathbb{1}_{\{\tau_n>0\}}X^-_{\tau_n\wedge t}|\mathcal{F}_s] = E[X^-_t|\mathcal{F}_s]$. This gives the supermartingale property. Also, the class DL property gives the integrability of X^-. Now the supermartingale property gives

$$E[X^+_t] \leq E[X_0] + E[X^-_t] < \infty.$$

Therefore X_t is integrable for each t. The adaptedness is clear.

Solution 3.3

(a) Denote by μ, ν the measures associated with f, g. Obviously we can write $f(T)g(T) - f(0)g(0) = \pi([0,T]\times[0,T])$ where π is the product measure induced by μ and ν on $[0,T]\times[0,T]$. Thus, using Fubini's theorem, we have

$$\begin{aligned}
f(T)g(T) &= \pi([0,T]\times[0,T]) \\
&= \int_0^T\int_0^T 1\,\mathrm{d}\pi + \pi(\{0\}\times(0,T]) + \pi((0,T]\times\{0\}) + \pi(\{(0,0)\}) \\
&= \int_0^T\int_0^T 1\,\mathrm{d}\pi + \pi(\{0\}\times(0,T]) + \pi((0,T]\times\{0\}) + f(0)g(0).
\end{aligned}$$

Note that

$$\begin{aligned}
\int_0^T\int_0^T 1\,\mathrm{d}\pi &= \int_0^T\int_0^T \mathbb{1}_{\{r<s\}}\mu(\mathrm{d}r)\nu(\mathrm{d}s) + \int_0^T\int_0^T \mathbb{1}_{\{r\geq s\}}\nu(\mathrm{d}s)\mu(\mathrm{d}r) \\
&= \int_0^T \mu((0,s))\nu(\mathrm{d}s) + \int_0^T \nu((0,r])\mu(\mathrm{d}r) \\
&= \int_0^T f(s-) - f(0)\,\mathrm{d}g(s) + \int_0^T g(s) - g(0)\,\mathrm{d}f(s) \\
&= \int_0^T f(s-))\,\mathrm{d}g(s) + \int_0^T g(s)\,\mathrm{d}f(s) - f(0)(g(T)-g(0)) - g(0)(f(T)-f(0)) \\
&= \int_0^T f(s-))\,\mathrm{d}g(s) + \int_0^T g(s)\,\mathrm{d}f(s) - \pi(\{0\}\times(0,T]) - \pi((0,T]\times\{0\}).
\end{aligned}$$

It follows that

$$f(T)g(T) = \int_0^T f(s-))\,\mathrm{d}g(s) + \int_0^T g(s)\,\mathrm{d}f(s) + f(0)g(0).$$

Of course, we can also derive more symmetrically that

$$\begin{aligned}
\int_0^T \int_0^T 1 \,\mathrm{d}\pi &= \int_0^T \int_0^T \mathbb{1}_{\{r<s\}} \mu(\mathrm{d}r)\nu(\mathrm{d}s) + \int_0^T \int_0^T \mathbb{1}_{\{r>s\}} \nu(\mathrm{d}s)\mu(\mathrm{d}r) \\
&\quad + \int \mathbb{1}_{\{r=s\}} \mu(\mathrm{d}r)\nu(\mathrm{d}s) \\
&= \int_0^T \mu((0,s))\nu(\mathrm{d}s) + \int_0^T \nu((0,r))\mu(\mathrm{d}r) + \int_0^T \mu(\{s\})\nu(\mathrm{d}s) \\
&= \int_0^T f(s-)\,\mathrm{d}g(s) + \int_0^T g(s-)\,\mathrm{d}f(s) + \sum_{0<s\leq T} \triangle f(s) \triangle g(s) \\
&\quad - \pi(\{0\} \times (0,T]) - \pi((0,T] \times \{0\}),
\end{aligned}$$

where we used that $\mu(\{s\})$ is nonzero iff $\triangle f(s)$ is nonzero, in which case the integral reduces to a sum. The rest follows exactly as above.

(b) First note that the collection $\mathcal{C} := \{X : X_t \text{ is } \mathcal{F}_{t-}\text{-measurable}\}$ is a vector space closed under multiplication and monotone bounded convergence. Also $\mathcal{C}$ contains the constant process 1. If X is adapted and left-continuous, then for each t, $X_t = \lim_{n\to\infty} X_{s_n}$ for any sequence $(s_n)_{n\in\mathbb{N}}$ with $s_n \uparrow t$. Note that each X_{s_n} is $\mathcal{F}_{t-}$-measurable. Therefore, using the completeness of the filtration, we get that X_t is $\mathcal{F}_{t-}$-measurable. Now by the monotone class theorem, $\mathcal{C}$ contains all bounded predictable processes. For a general predictable process X, we have $X \in \mathcal{C}$ because $X = \lim_{n\to\infty} X \wedge n \vee (-n)$ and each $X \wedge n \vee (-n)$ is in $\mathcal{C}$.

(c) For any partition π of $[0,T]$, write

$$M_T A_T = \sum_{i=1}^n M_T (A_{t_i} - A_{t_{i-1}}).$$

Because A is predictable, A_{t_i} is $\mathcal{F}_{t_i-}$-measurable, and because M is a martingale, we get

$$E[M_T A_T] = E\bigg[\sum_{i=1}^n M_{t_i-}(A_{t_i} - A_{t_{i-1}})\bigg].$$

As $|\pi| \to 0$, the sum inside the expectation converges to $\int_0^T M_{s-}\,\mathrm{d}A_s$, and because M is bounded and A is increasing and integrable, a majorant for all sums is $\|M\|_\infty A_T \in L^1$. So dominated convergence gives

$$\begin{aligned}
E[M_T A_T] &= \lim_{|\pi|\to 0} E\bigg[\sum_{t_i\in\pi} M_{t_i-}(A_{t_i} - A_{t_{i-1}})\bigg] \\
&= E\bigg[\lim_{|\pi|\to 0} \sum_{t_i\in\pi} M_{t_i-}(A_{t_i} - A_{t_{i-1}})\bigg] \\
&= E\bigg[\int_0^T M_{s-}\,\mathrm{d}A_s\bigg].
\end{aligned}$$

Solution 3.4

(a) Obviously, if $M \in \mathcal{H}_0^1$, then for any stopping time τ, it holds that $|M_\tau| \le M_T^*$. Then since $M_T^* \in L^1$, we must have

$$\limsup_{K\to\infty} \mathbb{E}[|M_\tau| \mathbb{1}_{\{|M_\tau|\ge K\}}] \le \lim_{K\to\infty} \mathbb{E}[|M_T^*| \mathbb{1}_{\{|M_T^*|\ge K\}}] = 0,$$

whence the uniform integrability of the family $\{M_\tau : \tau \text{ stopping time}\}$.

(b) Suppose that M is a local martingale in $\mathcal{H}_0^1$, and let $(\tau_n)_{n\ge1}$ be a sequence of stopping times such that $\mathbb{P}[\tau_n = T]$ tends to 1 and every stopped process M^{τ_n} is a martingale. The latter means that for any $0 \le s < t \le T$ and $A \in \mathcal{F}_s$, it holds that

$$\mathbb{E}[M_t^{\tau_n} \mathbb{1}_A] = \mathbb{E}[M_s^{\tau_n} \mathbb{1}_A].$$

Note that $|M_r^{\tau_n}| \le M_T^*$ holds for all $n \ge 1$ and for all $r \in [0,T]$, and because $M \in \mathcal{H}_0^1$ implies that $M_T^* \in L^1$, we can apply the dominated convergence theorem to both sides of the equation above and use the fact that $\lim_{n\to\infty} M_r^{\tau_n} = M_r$ for all $r \in [0,T]$ to obtain

$$\mathbb{E}[M_t \mathbb{1}_A] = \mathbb{E}[M_s \mathbb{1}_A],$$

which implies that M is in fact a martingale.

(c) Let $X = (X_t)_{t\in[0,T]}$ be a local martingale. Then there exists an increasing sequence of stopping times $(\tau_n)_{n\ge1}$ such that $\mathbb{P}[\tau_n = T]$ tends to 1 and for each n, the stopped process X^{τ_n} is a martingale. Since τ_n is bounded by a finite T, by the optional stopping theorem, X^{τ_n} is even a uniformly integrable martingale and therefore $X_\sigma^{\tau_n} = X_{\tau_n\wedge\sigma}$ is integrable for any stopping time σ. Now we define another sequence of stopping times $(\sigma_n)_{n\ge1}$ by

$$\sigma_n := \inf\Big\{t \ge 0 : |X_t| > n\Big\} \wedge T.$$

Clearly, $(\sigma_n)_{n\ge1}$ is increasing and satisfies that $\lim_{n\to\infty} \mathbb{P}[\sigma_n = T] = 1$. Moreover, by definition we have for each n that $|X_t| \le n$ for all $t < \sigma_n$ and therefore

$$|X_{\sigma_n-}| = |\lim_{t\to\sigma_n, t<\sigma_n} X_t| \le n.$$

As a result, for each n we obtain that

$$\begin{aligned}(X^{\tau_n\wedge\sigma_n})_T^* = \sup_{t\in[0,T]} |X_t^{\tau_n\wedge\sigma_n}| &\le \sup_{t<\tau_n\wedge\sigma_n} |X_t| + |X_{\tau_n\wedge\sigma_n}| \\ &\le n + |X_{\tau_n\wedge\sigma_n}|.\end{aligned}$$

Since $X_{\tau_n\wedge\sigma_n}$ is in L^1, the inequality above shows that $(X^{\tau_n\wedge\sigma_n})_T^*$ is in L^1 as well. On the other hand, by the optional stopping theorem it holds that $X^{\tau_n\wedge\sigma_n}$ is a local martingale null at time 0. Hence, we conclude that $X^{\tau_n\wedge\sigma_n} \in \mathcal{H}_0^1$ for all $n \ge 1$. Finally since $(\tau_n \wedge \sigma_n)_{n\ge1}$ is increasing and satisfies that $\lim_{n\to\infty} \mathbb{P}[\tau_n \wedge \sigma_n = T] = 1$, the claim follows.

Solution 3.5

(a) We define the stopping time $\rho+ := \inf\{t \in \mathcal{D}_n \,|\, t \geq \rho\}$. First, we observe that for each $t_i \in \mathcal{D}_n$

$$\mathbb{E}\Big[X^{\rho+}_{t_{i+1}} - X^{\rho+}_{t_i} \,\Big|\, \mathcal{F}_{t_i}\Big] = \mathbb{E}\Big[(X_{t_{i+1}} - X_{t_i})\, 1_{\{t_i<\rho\}} \,\Big|\, \mathcal{F}_{t_i}\Big] = 1_{\{t_i<\rho\}}\, \mathbb{E}\Big[X_{t_{i+1}} - X_{t_i} \,\Big|\, \mathcal{F}_{t_i}\Big].$$

Thus, we obtain that

$$\mathrm{MV}(X^{\rho+}, \mathcal{D}_n) := \mathbb{E}\Big[\sum_{t_i\in\mathcal{D}_n} \Big|\mathbb{E}[X^{\rho+}_{t_{i+1}} - X^{\rho+}_{t_i} \,|\, \mathcal{F}_{t_i}]\Big|\Big] = \sum_{t_i\in\mathcal{D}_n} \mathbb{E}\Big[1_{\{t_i<\rho\}} \Big|\mathbb{E}[X_{t_{i+1}} - X_{t_i} \,|\, \mathcal{F}_{t_i}]\Big|\Big].$$

By Jensen's inequality, we obtain for any two processes X' and X'' that

$$\begin{aligned}
\Big|\mathrm{MV}(X', \mathcal{D}_n) - \mathrm{MV}(X'', \mathcal{D}_n)\Big| &= \Big|\mathbb{E}[\sum_{t_i\in\mathcal{D}_n} \Big|\mathbb{E}[X'_{t_{i+1}} - X'_{t_i} \mid \mathcal{F}_{t_i}]\Big| - \Big|\mathbb{E}[X''_{t_{i+1}} - X''_{t_i} \mid \mathcal{F}_{t_i}]\Big|]\Big| \\
&\leq \mathbb{E}[\sum_{t_i\in\mathcal{D}_n} \Big|\Big|\mathbb{E}[X'_{t_{i+1}} - X'_{t_i} \mid \mathcal{F}_{t_i}]\Big| - \Big|\mathbb{E}[X''_{t_{i+1}} - X''_{t_i} \mid \mathcal{F}_{t_i}]\Big|\Big|] \\
&\leq \mathbb{E}[\sum_{t_i\in\mathcal{D}_n} \Big|\mathbb{E}[X'_{t_{i+1}} - X'_{t_i} \mid \mathcal{F}_{t_i}] - \mathbb{E}[X''_{t_{i+1}} - X''_{t_i} \mid \mathcal{F}_{t_i}]\Big|] \\
&\leq \mathbb{E}[\sum_{t_i\in\mathcal{D}_n} \mathbb{E}[\Big|(X'_{t_{i+1}} - X'_{t_i}) - (X''_{t_{i+1}} - X''_{t_i})\Big| \mid \mathcal{F}_{t_i}]] \\
&\leq \mathbb{E}\Big[\sum_{t_i\in\mathcal{D}_n} \Big|(X'_{t_{i+1}} - X'_{t_i}) - (X''_{t_{i+1}} - X''_{t_i})\Big|\Big].
\end{aligned}$$

Take $X' := X^\rho$ and $X'' := X^{\rho+}$. Then, we see that the only (possibly) non-zero term above in the sum is the one for which $\rho \in [t_i, t_{i+1})$. Thus, we obtain that

$$\Big|\mathrm{MV}(X^\rho, \mathcal{D}_n) - \mathrm{MV}(X^{\rho+}, \mathcal{D}_n)\Big| \leq 2\|X\|_\infty.$$

Remark: In fact, this holds true for any partition π of $[0, T]$.

(b) Let $n \in \mathbb{N}$ and let $0 = t_0 < \cdots < t_n = T$ be a finite partition of $[0, T]$. We have for all $i := 0, ..., n-1$ the existence of a sequence $(k_i^m)_m$ such that for each m, we have $k_i^m \in \mathcal{D}_m$, $k_i^m \geq k_{i+1}^m$, $k_i^m \geq t_i$ and $\lim_{m\to\infty} k_i^m = t_i$. Set $k_n^m := T$ for each m. Then we have for each m

$$\begin{aligned}
\mathrm{MV}(X, \pi) &= \mathbb{E}\Big[\sum_{i=0}^{n-1} \Big|\mathbb{E}[X_{t_{i+1}} - X_{t_i} \,|\, \mathcal{F}_{t_i}]\Big|\Big] \\
&\leq \mathbb{E}\Big[\sum_{i=0}^{n-1} \Big|\mathbb{E}[X_{k_{i+1}^m} - X_{k_i^m} \,|\, \mathcal{F}_{k_i^m}]\Big| + \Big|\mathbb{E}[X_{t_{i+1}} - X_{k_{i+1}^m} \,|\, \mathcal{F}_{t_{i+1}}]\Big| + \Big|\mathbb{E}[X_{k_i^m} - X_{t_i} \,|\, \mathcal{F}_{t_i}]\Big|\Big] \\
&\leq \mathbb{E}\Big[\sum_{j=0}^{2^m-1} \Big|\mathbb{E}[X_{jT/2^m} - X_{(j-1)T/2^m} \,|\mathcal{F}_{(j-1)T/2^m}]\Big|\Big] \\
&\quad + \mathbb{E}\Big[\sum_{i=0}^{n-1} \Big|\mathbb{E}[X_{t_{i+1}} - X_{k_{i+1}^m} \,|\, \mathcal{F}_{t_{i+1}}]\Big| + \Big|\mathbb{E}[X_{k_i^m} - X_{t_i} \,|\, \mathcal{F}_{t_i}]\Big|\Big]. \\
&= \mathrm{MV}(X, \mathcal{D}_m) + \mathbb{E}\Big[\sum_{i=0}^{n-1} \Big|\mathbb{E}[X_{t_{i+1}} - X_{k_{i+1}^m} \,|\, \mathcal{F}_{t_{i+1}}]\Big| + \Big|\mathbb{E}[X_{k_i^m} - X_{t_i} \,|\, \mathcal{F}_{t_i}]\Big|\Big].
\end{aligned}$$

By dominated convergence, as X is bounded and right-continuous, we have

$$\lim_{m\to\infty} \mathbb{E}\Big[\sum_{i=0}^{n-1} \Big|\mathbb{E}[X_{t_{i+1}} - X_{k_{i+1}^m} \mid \mathcal{F}_{t_{i+1}}]\Big| + \Big|\mathbb{E}[X_{k_i^m} - X_{t_i} \mid \mathcal{F}_{t_i}]\Big|\Big] = 0.$$

Thus, we obtain that

$$\mathrm{MV}(X,\pi) \le \lim_{m\to\infty} \mathrm{MV}(X,\mathcal{D}_m).$$

As the partition was arbitrarily chosen, taking the sup over all the finite partitions in the above inequality yields that $\mathrm{MV}(X) \le \lim_{m\to\infty} \mathrm{MV}(X,\mathcal{D}_m)$, and so $\mathrm{MV}(X) = \lim_{m\to\infty} \mathrm{MV}(X,\mathcal{D}_m)$.