Mathematical Finance

Solution sheet 4

Solution 4.1

(a) First, $E[1\wedge|Z^1-Z^2|]\geq 0$ for all $Z^1,Z^2\in L^0$ and $E[1\wedge|Z^1-Z^2|]=0$ if and only if $Z^1=Z^2$ P-a.s. Also $E[1\wedge|Z^1-Z^2|]=E[1\wedge|Z^2-Z^1|]$ gives the symmetry. Finally, for $Z^1,Z^2,Z^3\in L^0$, we have $1\wedge|Z^1-Z^3|\leq 1\wedge(|Z^1-Z^2|+|Z^2-Z^3|)\leq 1\wedge|Z^1-Z^2|+1\wedge|Z^2-Z^3|$. Taking expectation gives the triangle inequality.

The estimate $P[|Z|>\delta]\leq\delta^{-1}E[1\wedge|Z|]$for $\delta\leq 1$ gives that convergence in d_0 implies convergence in probability. If $Z^n\to Z$ in probability, then $E[1\wedge|Z^n-Z|]\to 0$ by the bounded convergence theorem.

(b) We only show that $(\mathbb{L},d)$ is a complete metric space. The proof for $(\mathbb{D},d)$ is analogous. The fact that d is a metric can be proved similarly as above. Now let $(X^n)_{n\in\mathbb{N}}\subseteq\mathbb{L}$ be a Cauchy sequence in d. Set $n_1=1$. After n_{k-1} is defined, we choose n_k such that $P[(X^n-X^m)^*_T>2^{-k}]<2^{-k}$ for all $m,n\geq n_k$. In particular, this gives

$$\sum_{k=1}^{\infty}P[(X^{n_{k+1}}-X^{n_k})^*_T>2^{-k}]<\infty$$

and by the Borel-Cantelli lemma it follows that $(X^{n_k})_{k\in\mathbb{N}}$ is P-a.s. a Cauchy sequence under the uniform convergence on $[0,T]$. So for each $t\in[0,T]$ there is a limit $X_t=\lim_{k\to\infty}X^{n_k}_t$. Since each X^{n_k} is adapted and $\mathbb{F}$ is complete, X is also adapted. Using $\sup_{s\in[0,T)}|X^{n_k}_{s+}-X^{n_l}_{s+}|\leq\sup_{s\in[0,T]}|X^{n_k}_s-X^{n_l}_s|$, we have $(X^{n_k}_+)_{k\in\mathbb{N}}$ is also Cauchy under uniform convergence on $[0,T]$ and has a limit X_+. Using the uniform convergence on $[0,T]$, we can switch the following limits and obtain

$$\lim_{s\uparrow t}X_s=\lim_{s\uparrow t}\lim_{k\to\infty}X^{n_k}_s=\lim_{k\to\infty}\lim_{s\uparrow t}X^{n_k}_s=X_t,$$
$$\lim_{s\downarrow t}X_s=\lim_{s\downarrow t}\lim_{k\to\infty}X^{n_k}_s=\lim_{k\to\infty}\lim_{s\downarrow t}X^{n_k}_s=X_{t+}.$$

Therefore $X\in\mathbb{L}$. Finally, we need to show that $(X^n)_{n\in\mathbb{N}}$ converges to X in d. Let $\varepsilon>0$. Find k such that $d(X^{n_k},X)<\varepsilon/2$ and $1/2^k<\varepsilon/2$. Then for all $n\geq n_k$, we have $d(X^n,X)\leq d(X^n,X^{n_k})+d(X^{n_k},X)<\varepsilon$. This shows $d(X^n,X)\to 0$ as $n\to\infty$.

(c) Let $X\in\mathbb{L}$. By localization, we may assume that X is bounded. Let $\varepsilon>0$. Set $Y=X_+$ and consider $\tau^\varepsilon_0:=0$ and $\tau^\varepsilon_{n+1}:=\inf\{t>\tau^\varepsilon_n:|Y_t-Y_{\tau^\varepsilon_n}|>\varepsilon\}\wedge T$.

The right-continuity of Y implies that the τ_n^ε are stopping times and $\tau_n^\varepsilon \uparrow T$ stationarily. So define $X^\varepsilon = X_0 \mathbb{1}_{\{0\}} + \sum_n Y_{\tau_n^\varepsilon} \mathbb{1}_{]\!]\tau_n^\varepsilon, \tau_{n+1}^\varepsilon]\!]} \in b\mathcal{E}_0$ and by construction $P[(X^\varepsilon - X)_T^* > \varepsilon] \leq P[\tau_m \neq T] + P[(X^\varepsilon - X^{\tau_m})_T^* > \varepsilon]$. Observe $Y_- = X$ and the result follows.

Solution 4.2

(a) "$\Longrightarrow$" Let $\varepsilon > 0$. By definition there exists $K > 0$ with $\sup_n P[|c_n| > K] \leq \varepsilon$. Then

$$P[|\lambda_n c_n| > \varepsilon] \leq \sup_n P[|c_n| > K] + P[\{|\lambda_n c_n| > \varepsilon\} \cap \{|c_n| \leq K\}].$$

The second term goes to 0 because $\lambda_n \downarrow 0$. Therefore, we have $\lambda_n c_n \to 0$ in L^0.

"$\Longleftarrow$" If C is not bounded in L^0, then there exist δ and a sequence (c_n) such that $P[|c_n| > n] > \delta$ for all $n \in \mathbb{N}$. It follows that $P[|c_n/n| > 1] > \delta$ for all $n \in \mathbb{N}$, so $|c_n/n| \not\to 0$ in L^0. This is a contradiction.

(b) "$\Longrightarrow$" If the set is not bounded in L^0, then there exist $\delta > 0$ and a sequence $(H^n) \subseteq b\mathcal{E}$ with $\|H^n\|_\infty \leq 1$ such that $P[|H^n \bullet X| > n] > \delta$ for all $n \in \mathbb{N}$. Then $\tilde{H}^n := H^n/n \to 0$ uniformly in (ω, t) but $P[|\tilde{H}^n \bullet X| > 1] = P[|H^n \bullet X| > n] > \delta$ for all $n \in \mathbb{N}$. This is a contradiction.

"$\Longleftarrow$" Let $(H^n) \subseteq b\mathcal{E}$ with $\|H^n\|_\infty \to 0$. WLOG, we assume that $\|H^n\|_\infty > 0$ for all n. Set $\tilde{H}^n = H^n/\|H^n\|_\infty$. Then $\{\tilde{H}^n \bullet X_T : n \in \mathbb{N}\}$ is bounded in L^0 and therefore by part (a) with $\lambda_n = \|H^n\|_\infty$, $H^n \bullet X_T = \lambda_n \tilde{H}^n \bullet X_T \to 0$ in L^0. This shows that X is a good integrator.

Solution 4.3

(a) We only need to prove that if each X^n is a local submartingale, then X is locally a submartingale. By passing to a subsequence, we may assume that $X^n \to X$ uniformly on $[0, T]$ P-a.s. By Exercise 4.1, we have $X \in \mathbb{D}$. Moreover, $M_t := \sup_n (X^n)_t^*$ is RCLL, adapted and increasing. Observe that $|\triangle M| \leq \sup_n |\triangle X^n|$, which is locally integrable by assumption. So M is locally integrable. Let $(\tau_k)_{k \in \mathbb{N}}$ be a localizing sequence so that $\mathbb{1}_{\{\tau_k > 0\}} M^{\tau_k}$ is integrable. For each $n \in \mathbb{N}$, the process $\mathbb{1}_{\{\tau_k > 0\}} (X^n)^{\tau_k}$ is locally a submartingale. Moreover $\mathbb{1}_{\{\tau_k > 0\}} |X^n|^{\tau_k} \leq \mathbb{1}_{\{\tau_k > 0\}} M^{\tau_k}$, so $\mathbb{1}_{\{\tau_k > 0\}} |X^n|^{\tau_k}$ is of class DL. So by Exercise 3.2 (c), for every $n \in \mathbb{N}$, the process $\mathbb{1}_{\{\tau_k > 0\}} (X^n)^{\tau_k}$ is a submartingale. Now applying the dominated convergence, we can send $n \to \infty$ and get that $\mathbb{1}_{\{\tau_k > 0\}} X^{\tau_k}$ is a submartingale.

(b) By localization, we may assume that $H \in b\mathcal{P}$ and $M \in \mathcal{H}_0^1$. Suppose $\|H\|_\infty \leq K$ for some constant $K > 0$. Choose a sequence $(H^n)_{n \in \mathbb{N}} \subseteq b\mathcal{E}$ with $d_E(H^n \bullet M, H \bullet M) \to 0$. In particular, $d(H^n \bullet M, H \bullet M) \to 0$. Note that each $Y^n := H^n \bullet M$ is a martingale and moreover $|\triangle Y^n| = |H^n \triangle X| \leq K |\triangle X|$.

Therefore $\sup_n |\triangle Y^n| \leq K|\triangle X|$. Because $X \in \mathcal{H}_0^1$, we have that $|\triangle X|$ is integrable and so is $\sup_n |\triangle Y^n|$. Now applying part (a), we conclude that $Y = H \bullet M$ is a martingale.

Solution 4.4

(a) We set $P := X + Y$ and $D := X - Y$. Then

$$\begin{aligned} XY &= \frac{1}{4}(P^2 - D^2) \\ &= \frac{1}{4}\Big(P_0^2 + 2\int P_- \,\mathrm{d}P + [P] - D_0^2 - 2\int D_- \,\mathrm{d}D - [D]\Big) \\ &= \frac{1}{4}\Big(4X_0Y_0 + 4\int X_- \,\mathrm{d}Y + 4\int Y_- \,\mathrm{d}X + [P] - [D]\Big) \\ &= X_0Y_0 + \int X_- \,\mathrm{d}Y + \int Y_- \,\mathrm{d}X + [X, Y]. \end{aligned}$$

(b) Using the integration by parts formula, we have

$$\triangle[X, Y] = \triangle(XY) - X_-\triangle Y - Y_-\triangle X = \triangle X \triangle Y.$$

(c) Recall that $t \mapsto [X]_t$ is a.s. increasing, so we must have

$$\sum_{s \leq t} (\triangle X_s)^2 = \sum_{s \leq t} \triangle [X]_s \leq [X]_t < \infty.$$

(d) We express the covariation as the limit along equally spaced partitions of $[0, T]$. This gives

$$\begin{aligned} [X, V]_T &= \lim_{n\to\infty} \sum_{k=1}^{n} (X_{kT/n} - X_{(k-1)T/n})(V_{kT/n} - V_{(k-1)T/n}) \\ &= \lim_{n\to\infty} \sum_{k=1}^{n} \int \mathbb{1}_{\{(k-1)T/n < s \leq kT/n\}} (X_{kT/n} - X_{(k-1)T/n}) \,\mathrm{d}V_s \\ &= \int \lim_{n\to\infty} \sum_{k=1}^{n} \mathbb{1}_{\{(k-1)T/n < s \leq kT/n\}} (X_{kT/n} - X_{(k-1)T/n}) \,\mathrm{d}V_s \\ &= \int_0^T \triangle X_s \,\mathrm{d}V_s. \end{aligned}$$

We used the bounded convergence theorem in the third line. Now, since X is RCLL, on each sample path there is only a countable set S of times at which $\triangle X \neq 0$. So we can use the bounded convergence theorem again to obtain

$$\begin{aligned} \int_0^T \triangle X_s \,\mathrm{d}V_s &= \int_0^T \sum_{s \in S} \triangle X_s \mathbb{1}_{\{s=r\}} \,\mathrm{d}V_r \\ &= \sum_{s \in S} \int_0^T \triangle X_s \mathbb{1}_{\{s=r\}} \,\mathrm{d}V_r \\ &= \sum_{s \in S, s \leq T} \triangle X_s \triangle V_s = \sum_{s \leq T} \triangle X_s \triangle V_s. \end{aligned}$$

Solution 4.5

(a) Let $(K^n)_{n\in\mathbb{N}} \subset b\mathcal{E}$ with $\|K^n\|_\infty \le 1$ for all $n \in \mathbb{N}$. Because $K^n \bullet (H^n \bullet A) = (K^n H^n)\bullet A$ and $K^n H^n \to 0$ with $\|K^n H^n\|_\infty \le \|H^n\|_\infty$, applying the dominated convergence theorem yields $(K^n H^n) \bullet A_T \to 0$ P-a.s. and hence in L^0. In view of Lemma 3.2, we conclude that $d'_E(H^n \bullet A, 0) \to 0$.

(b) Choose a localizing sequnce $(\tau_m)_{m\in\mathbb{N}}$ such that $M^{\tau_m} \in \mathcal{H}^1_0$ and $P[\tau_m = T] \uparrow 1$. The dominating convergence property in Theorem 3.12 gives that for every $m \in \mathbb{N}$, $(H^n \bullet M^{\tau_m})_{n\in\mathbb{N}}$ is Cauchy in $(\mathcal{S}, d'_E)$. Set $Y = H^i - H^j$. Using that $H^n \bullet M^{\tau_m} = (H^n \bullet M)^{\tau_m}$ and choosing ε such that $P[\tau_m \ne T] < \varepsilon$ for large m, we have for all $(K^n) \subseteq b\mathcal{E}$ with $\|K^n\|_\infty \le 1$ that

$$P[(K^n Y) \bullet M_T > \varepsilon] \le P[\tau_m \ne T] + P[(K^n Y) \bullet M_{\tau_m} > \varepsilon].$$

In view of Lemma 3.2, we conclude that $(H^n \bullet M)_{n\in\mathbb{N}}$ is Cauchy in $(\mathcal{S}, d'_E)$.

(c) Assume that $S = S_0 + M + A$ where $M \in \mathcal{M}_{0,\text{loc}}$ and A is of FV. We first construct $H \bullet M$. Choose a localizing sequence $(\tau_m)_{m\in\mathbb{N}}$ such that $P[\tau_m = T] \uparrow 1$ and $H^m := H\mathbb{1}_{[\![0,\tau_m]\!]} \in b\mathcal{P}$. Part (b) gives that $H^m \bullet M$ exists. So we can define $H \bullet M := H^m \bullet M$ on $[\![0, \tau_m]\!]$. To see this definition is consistent, suppose that $(\sigma_n)_{n\in\mathbb{N}}$ is another localizing sequence. It follows that $H \bullet M = H\mathbb{1}_{[\![0,\tau_m\wedge\sigma_n]\!]} \bullet M$ on $[\![0, \tau_n \wedge \sigma_m]\!]$. So for large enough n, it is clear that $H \bullet M = H\mathbb{1}_{[\![0,\tau_m\wedge\sigma_n]\!]} \bullet M = H^m \bullet M$ on $[\![0, \tau_m]\!]$. This implies that the definition is independent of the choice of localizing sequences. Using part (a), we can analogously construct $H \bullet A$ and hence $H \bullet S := H \bullet M + H \bullet A$.