Mathematical Finance

Solution sheet 5

Solution 5.1 Define the process

$$S_t = \begin{cases} 0 & \text{for } 0 \leq t < T, \\ X & \text{for } t = T, \end{cases}$$

where X is normally distributed with mean $\mu = 0$ and variance $\sigma^2 > 0$. Use the natural filtration $\mathbb{F}$ of S. Obviously, S is a martingale. Let ϑ be predictable w.r.t $\mathbb{F}$. As we saw in Exercise 1.1 (c) and Exercise 3.3 (b), $\vartheta \equiv c$ for some $c \in \mathbb{R}$ on $[0, T]$. Therefore $\vartheta \bullet S_t = 0$ if $t < T$ and $\vartheta \bullet S_T = cX$. If $c \neq 0$ and $cX \geq -a$ P-a.s. for some $a \in \mathbb{R}$, then by symmetry $cX \leq a$ P-a.s. which is a contradiction.
Remark: The same reasoning works for any distribution with mean 0 and unbounded support on both sides.

Solution 5.2

(a) First, assume that S is bounded. Note that then every simple strategy is admissible. Moreover, S is a uniformly integrable Q-martingale if and only if $E_Q[S_\tau - S_0] = 0$ for all stopping times (taking values in $[0, T]$). So let τ be an arbitrary stopping time, and consider the simple strategies $\vartheta^\pm := \pm 1_{]\!]0,\tau]\!]}$. Using that Q is an equivalent separating measure for S then gives

$$0 \geq E_Q[\vartheta^\pm \bullet S_T] = \pm E_Q[S_\tau - S_0]. \tag{1}$$

If S is locally bounded, then there exists an increasing sequence of stopping times $(\sigma_n)_{n\in\mathbb{N}}$ taking values in $[0, T]$ with $\lim_{n\to\infty} P[\sigma_n = T] = 1$ such that S^{σ_n} is bounded for all $n \in \mathbb{N}$. It suffices to show that for each $n \in \mathbb{N}$, S^{σ_n} is a uniformly integrable Q-martingale. To this end, fix $n \in \mathbb{N}$. It suffices to show that for each stopping time τ with $\tau \leq \sigma_n$ P-a.s., $E_Q[S_\tau - S_0] = 0$. So let τ be such a stopping time, and consider as above the simple strategies $\vartheta^\pm := \pm 1_{]\!]0,\tau]\!]}$. Then both strategies are admissible since S is bounded on $[\![0, \sigma_n]\!]$ and $\tau \leq \sigma_n$ P-a.s., and the same argument as in the first step gives $E_Q[S_\tau - S_0] = 0$.

(b) By assumption, there exist a strictly positive predictable process $\psi = (\psi_t)_{t\in[0,T]}$, an $\mathbb{R}^d$-valued (local) Q-martingale M and an $\mathbb{R}^d$-valued $\mathcal{F}_0$-measurable random vector S_0 such that $S = S_0 + \psi \bullet M$. Let $\vartheta \in \Theta_{\text{adm}}$. Then by the associativity of the stochastic integral, $G(\vartheta) = \vartheta \bullet S = (\vartheta\psi) \bullet M$. Moreover, as $(\vartheta\psi) \bullet M$ is uniformly bounded from below by admissibility, it is a local Q-martingale by the

Ansel-Stricker theorem. By Fatou's lemma, it is then also a Q-supermartingale, and hence

$$E_Q[G_T(\vartheta)] \leq E_Q[G_0(\vartheta)] = 0. \tag{2}$$

(c) Let $\vartheta \in L(S)$ be arbitrary. Then

$$G_T(\vartheta) = \vartheta \bullet S_T = \vartheta \bullet S_{T-} + \vartheta_T \Delta S_T = \lim_{t \uparrow T} \vartheta \bullet S_t + \vartheta_T X = 0 + \vartheta_T X = \vartheta_T X. \tag{3}$$

As we saw in the solution of Exercise 5.1, ϑ_T is deterministic and $\vartheta \in \Theta_{\text{adm}}$ if and only if $\vartheta_T = 0$. Thus, we may conclude that $G_T(\vartheta) = 0$ for all $\vartheta \in \Theta_{\text{adm}}$. Therefore the condition

$$E_Q[G_T(\vartheta)] \leq 0 \quad \text{for all } \vartheta \in \Theta_{\text{adm}}$$

is trivially satisfied for each probability measure $Q \approx P$ on $\mathcal{F}_T$. In particular, P itself is a separating measure.

Finally if $Q \approx P$ on $\mathcal{F}_T$ is an equivalent probability measure, by the first step (whose results remain unchanged by an equivalent change of measure), $M = (M_t)_{t \in [0,T]}$ is a Q-martingale null at 0 for the filtration $(\mathcal{F}_t)_{t \in [0,T]}$ if and only if M_T is $\sigma(X)$-measurable, Q-integrable with mean 0 and $M_t = 0$ for all $t \in [0, T)$. Moreover, if $\psi \in L^Q(M)$, then as M is constant and equal to 0 on $[0, T)$,

$$\psi \bullet M_t = \begin{cases} 0 & \text{for } t < T, \\ \psi_T M_T & \text{for } t = T. \end{cases} \tag{4}$$

Note that as ψ_T is constant, $\psi \bullet M$ is a true Q-martingale, and thus Q is an equivalent σ-martingale measure for S if and only if it is an equivalent martingale measure. Since $E[S_T] = \mu \neq 0$, P is not a martingale measure and hence also not a σ-martingale measure.

Solution 5.3

(a) It is enough to treat the 1-dimensional case. Since X is a σ-martingale with $X_0 = 0$, there exist a local martingale M and a positive predictable integrand ψ in $L(M)$ such that $X = \psi \bullet M$. For each $n \geq 1$, we define $A_n := \{|\psi| \leq n\}$. Since 1_{A_n} is predictable and bounded, it is X-integrable and therefore by the associativity of stochastic integrals, $1_{A_n}\psi$ is M-integrable. In particular, since $\psi^n := 1_{A_n}\psi$ is bounded by n, the stochastic integral $1_{A_n} \bullet X = \psi^n \bullet M$ is a local martingale. Then, for each fixed n, since every local martingale starting at null is locally an $\mathcal{H}_0^1$-martingale and therefore uniformly integrable, one can find a sequence of stopping times $(\tau_m^n)_{m \geq 1}$ such that τ_m^n increases to ∞ and for each m the stopped process $(1_{A_n} \bullet X)^{\tau_m^n}$ is an $\mathcal{H}_0^1$-martingale. Now we define $\Sigma_n := A_n \cap [\![0, \max_{k \leq n} \tau_n^k]\!]$ for each $n \geq 1$. Then using the associativity of stochastic integrals, we obtain that $1_{\Sigma_n} \bullet X = (1_{A_n} \bullet X)^{\tau_m^n}$ is a uniformly integrable martingale for each n. Finally, noting that by definition, each Σ_n is predictable, $\Sigma_n \subseteq \Sigma_{n+1}$ holds for all n and $\bigcup_n \Sigma_n = \Omega \times [0, \infty)$, we complete the proof.

(b) For notational simplicity, we assume $d = 1$. Let X be a continuous σ-martingale (null at 0) so that $X = \psi \bullet M$ for a local martingale M and a positive predictable integrand $\psi \in L(M)$. Then by continuity, X is locally bounded in the sense that there exists a sequence of stopping times (τ_n) such that τ_n increases to infinity and each X^{τ_n} is bounded; for instance we can take $\tau_n := \inf\{t > 0 \mid |X_t| \geq n\}$. Note that for each n, we have $X^{\tau_n} = 1_{[\![0,\tau_n]\!]} \bullet X = 1_{[\![0,\tau_n]\!]} \bullet (\psi \bullet M) = (1_{[\![0,\tau_n]\!]}\psi) \bullet M$. Now the boundedness of X^{τ_n} and the Ansel–Stricker argument guarantee that $X^{\tau_n} = (1_{[\![0,\tau_n]\!]}\psi) \bullet M$ is a martingale. Hence X is a local martingale.
In fact we can use the same argument to show that any σ-martingale which is locally bounded from below (or from above) is a local martingale.