Mathematical Finance

Solution sheet 6

Solution 6.1

(a) Take $U = \mathbb{R}_{++} = (0, \infty)$ and $f : x \mapsto 1/x$. Because $Y_- > 0$, the processes $f'(Y_-) = -1/Y_-^2$ and $f''(Y_-) = 2/Y_-^3$ are predictable and locally bounded. Hence, $f(Y_-) \bullet Y$ and $f''(Y_-) \bullet [Y]$ are semimartingales. Clearly, the sums of the jump terms are also semimartingales. Thus, by Itô's formula, $1/Y = f(Y)$ is a semimartingale.

(b) Suppose $X, Y \in \mathcal{X}_{++}^1$ are numéraire portfolios. Both X/Y and Y/X are supermartingales starting from 1 at time 0. Set $R := X/Y$. Then $E[R_t] \le E[R_0] = 1$ for all $t > 0$. On the other hand, $E[1/S_t] \le E[1/R_0] = 1$ for all $t > 0$. But by Jensen's inequality $E[1/R_t] \ge 1/E[R_t] > 1$ and this gives $E[1/R_t] = 1$ and $E[R_t] = 1$ for all $t > 0$. Note that $x \mapsto 1/x$ is strictly convex on $(0, \infty)$, so this can happen if and only if $S_t = 1$, i.e. $X_t = Y_t$ P-a.s. for all $t > 0$.

Solution 6.2

(a) By definition of being a σ-martingale, $ZS = \int \psi \, dM$ for a local martingale M and an integrand $\psi \in L(M)$ with $\psi > 0$. Recall that $\mathcal{M}_{0,\mathrm{loc}} = \mathcal{H}^1_{0,\mathrm{loc}}$.

Now, let (ρ_n) be a localizing sequence for M such that $M^{\rho_n} \in \mathcal{H}^1_0$ for each n. As in the proof of the Ansel–Stricker theorem (Lemma 4.2 in the lecture notes), for each k, we define $\psi^k := \psi \, 1_{\{|\psi| \le k\}}$ and for each n and k define $M^{n,k} := \int \psi^k \, dM^{\rho_n}$, which is in $\mathcal{H}^1_0$.

By definition, $M^{n,k} \to \int \psi \, dM^{\rho_n}$ as $k \to \infty$ for d_S and therefore for d for each fixed n, as well as $(\Delta M^{n,k})^\pm \le (\Delta \int \psi \, dM^{\rho_n})^\pm$. Assume for the moment that $\int \psi \, dM$ has locally a lower bound in L^1, which means that there exist a localizing sequence (τ_m) of stopping times and a sequence of random variables $(\gamma_m) \subseteq L^1$ such that $(\int \psi \, dM)^{\tau_m} \ge \gamma_m$ for each m. In particular, we have $(\int \psi \, dM^{\rho_n})^{\tau_m} \ge \gamma_m$ for each n, m. Thus, all the assumptions of Lemma 4.2 are satisfied for each n, which implies that $\int \psi \, dM^{\rho_n} = (\int \psi \, dM)^{\rho_n} \in \mathcal{M}_{0,\mathrm{loc}}$ for each n. This implies that $\int \psi \, dM \in \mathcal{M}_{0,\mathrm{loc}}$. So we have proved the claim. Therefore, it remains to show that $ZS = \int \psi \, dM$ has locally a lower bound in L^1.

Let (T_n) be a localizing sequence for the local martingale Z such that each Z^{T_n} is an $\mathcal{H}^1$-martingale and define for each $n \in \mathbb{N}$ the stopping times

$\sigma_n := \{t \geq 0 : |S_t| \geq n\}$ and $\widehat{T}_n := \{t \geq 0 : |Z_t| \geq n\}$. Consider the sequence of stopping times (τ_n) defined by $\tau_n := T_n \wedge \widehat{T}_n \wedge \sigma_n$. By construction, (τ_n) converges to ∞ a.s. Moreover, as S is continuous and

$$\sup_{t\geq 0} |Z_t^{\tau_n}| \leq n + |\Delta Z_{\tau_n}| \in L^1,$$

we obtain for each n that $(ZS)^{\tau_n} \geq -n(n + |\Delta Z_{\tau_n}|) \in L^1$.

(b) First let us prove the following result:

Let $Y = (Y_t)_{t\in[0,T]}$ be an adapted RCLL $\mathbb{R}^d$-valued process and $Q \approx P$ an equivalent measure with density process $Z_t := \frac{dQ}{dP}|_{\mathcal{F}_t}$. Then Y is a Q-σ-martingale if and only if ZY is a P-σ-martingale.

For simplicity, let $d = 1$ and $Y_0 = 0$. Suppose that Y is a Q-σ-martingale such that $Y = \phi \bullet M$ for some Q-local martingale M and some $\phi \in L(M)$ with $\phi > 0$. Using the product rule we have

$$d(ZY) = Y_- dZ + Z_- dY + d[Z, Y].$$

By the associativity of stochastic integrals, inserting $Y = \phi \bullet M$, we obtain that

$$Z_- dY = \phi Z_- dM = \phi d(Z_- \bullet M). \tag{1}$$

Moreover, we also have

$$d[Z, Y] = d[Z, (\phi \bullet M)] = \phi d[Z, M]. \tag{2}$$

Now we apply the product rule for ZM and obtain that
$d(ZM) = Z_0M_0 + Z_- dM + M_- dZ + d[M, Z]$, which implies that

$$Z_- \bullet M = ZM - Z_0M_0 - M_- \bullet Z - [M, Z]. \tag{3}$$

From (1) and (2) we know that $\phi \in L(Z_- \bullet M)$ and $\phi \in L([M, Z])$, therefore (3) implies that $\phi \in L(ZM - Z_0M_0 - M_- \bullet Z)$ and consequently we have

$$\begin{aligned} d(ZY) = Y_- dZ + Z_- dY + d[Z, Y] &= Y_- dZ + \phi d(ZM - Z_0M_0 - M_- \bullet Z) \\ &\quad - \phi d[Z, M] + \phi d[Z, M] \\ &= Y_- dZ + \phi d(ZM - Z_0M_0 - M_- \bullet Z). \end{aligned}$$

Clearly the density process Z is a P-martingale. Moreover, by Bayes' theorem, with M being a Q-local martingale the process ZM is a P-local martingale. Also, by the hint, the stochastic integral $M_- \bullet Z$ is a P-local martingale. Hence $ZM - Z_0M_0 - M_- \bullet Z$ is a P-local martingale and $\phi \bullet (ZM - Z_0M_0 - M_- \bullet Z)$ is a P-σ-martingale. Since $Y_- \bullet Z$ is a P-local martingale, as the sum of two P-σ-martingales, $ZY = Y_- \bullet Z + \phi d(ZM - Z_0M_0 - M_- \bullet Z)$ is also a P-σ-martingale. By symmetry (we replace Y by ZY and replace Z by $\frac{1}{Z}$ and note that $\frac{1}{Z}$ as the density process of $\frac{dP}{dQ}|_{\mathcal{F}_t}$ is a Q-martingale), we can use the same argument to show the converse.

Now assume that S has a P-EσMD, say, D, so that $D > 0$ is a P-local martingale and DS is a P-σ-martingale; and $Q \approx P$ on $\mathcal{F}_T$ with density process Z. Define $Y := \frac{Z_0}{Z} DS$. Obviously $ZY = Z_0 DS$ is a P-σ-martingale by assumption. Hence, using the above result, we can deduce that Y is a Q-σ-martingale. By definition we can conclude that $\frac{Z_0}{Z}D$ is a Q-EσMD for S.

(c) First write $X = 1 + G(\vartheta)$ for some $G(\vartheta) \geq -1$ We could compute.

$$
\begin{aligned}
ZG(\vartheta) &= G_-(\vartheta) \bullet Z + (Z_-\vartheta) \bullet S + [Z, G(\vartheta)] \\
&= G_-(\vartheta) \bullet Z + (\vartheta Z_-) \bullet S + \vartheta \bullet [Z, S] \\
&= G_-(\vartheta) \bullet Z + \vartheta \bullet (\underbrace{Z_- \bullet S + [Z, S] + S_- \bullet Z}_{=ZS} - S_- \bullet Z) \\
&= G_-(\vartheta) \bullet Z + \vartheta \bullet (ZS) - (S_-\vartheta) \bullet Z
\end{aligned}
$$

like ZS is a stochastic integral of some P-local martingale and so is $ZX = Z + ZG(\vartheta)$. Moreover, $ZX \geq 0$ and by Ansel-Stricker is a P-supermartingale.

Solution 6.3

(a) $d\langle B, W\rangle_t = \rho dt$ because $d\langle W, W'\rangle_t = 0$, and

$$
\begin{aligned}
\langle S, Y\rangle_t &= \left\langle \int \sigma(u, S_u, Y_u) dW_u, \int a(u, Y_u) dB_u \right\rangle_t \\
&= \int_0^t \sigma(u, S_u, Y_u) a(u, Y_u) d\langle W, B\rangle_u = \int_0^t \sigma(u, S_u, Y_u) a(u, Y_u) \rho du.
\end{aligned}
$$

(b) Let Z^Q be the density process of $Q \approx P$. Note that $\mathcal{F}_0$ is trivial and Z^Q is continuous since the filtration is generated by (W, W'). Defining L^Q by

$$
L^Q = \int \frac{1}{Z^Q} dZ^Q,
$$

we have $Z^Q = \mathcal{E}(L^Q)$. By the Kunita–Watanabe decomposition, L^Q is given by

$$
L^Q = \int \gamma^Q \sigma dW + N^Q
$$

with $N^Q \in \mathcal{M}_{0,\mathrm{loc}}(P)$ and $\left\langle N^Q, \int \sigma dW \right\rangle = 0$. By Bayes' rule, Q is an ELMM for S iff $Z^Q S \in \mathcal{M}_{\mathrm{loc}}(P)$. By the product rule, we obtain

$$
\begin{aligned}
d(Z_t^Q S_t) &= Z_t^Q \sigma_t dW_t + S_t dZ_t^Q + Z_t^Q \left(\mu_t dt + d\left\langle L^Q, \int \sigma dW \right\rangle_t \right) \\
&= Z_t^Q \sigma_t dW_t + S_t dZ_t^Q + Z_t^Q \left(\mu_t dt + \gamma_t^Q \sigma_t^2 dt \right),
\end{aligned}
$$

yielding $Z^Q S \in \mathcal{M}_{\text{loc}}(P)$ if and only if $\gamma_t^Q = -\frac{\mu_t}{\sigma_t^2}$. Therefore the equivalent local martingale measures Q are parametrized via

$$Z^Q = \mathcal{E}\left(-\int \frac{\mu}{\sigma} dW + N^Q\right).$$

Since the filtration is generated by (W, W'), we can apply the martingale representation theorem to write N^Q as

$$N^Q = \int \psi dW + \int \nu dW',$$

where ψ and ν are some predictable processes. As $\left\langle N^Q, \int \sigma dW \right\rangle = 0$, it follows that $0 = \int \psi_t \, \sigma_t \, dt$ and hence $\psi = 0$ so that we finally obtain

$$Z^Q = \mathcal{E}\left(-\int \lambda dW + \int \nu dW'\right), \tag{4}$$

where $\lambda = \mu/\sigma$ and ν is some predictable process.

(c) By Girsanov, (W^Q, W'^Q), defined by $W^Q = W + \int \lambda dt$ and $W'^Q = W' - \int \nu dt$, is a 2-dimensional Q-Brownian motion. Plugging this into the SDEs for S and Y gives

$$dS_t = \mu_t dt + \sigma_t (dW_t^Q - \lambda_t dt) = \sigma_t dW_t^Q$$

and

$$\begin{aligned} dY_t &= b_t dt + a_t \rho (dW_t^Q - \lambda_t dt) + a_t \sqrt{1-\rho^2} (dW_t'^Q + \nu_t dt) \\ &= \left(b_t + a_t (\sqrt{1-\rho^2}\ \nu_t - \rho \lambda_t)\right) dt + a_t dB^Q \end{aligned}$$

for the Q-Brownian motion $B_t^Q = \rho W_t^Q + \sqrt{1-\rho^2}\ W_t'^Q$.