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Mathematical Finance

Solution sheet 7

Solution 7.1

(a) Since Z is an exponential Lévy process, it is a P-martingale if and only if it is integrable with mean 1. First, note that

$$E[\exp(\phi(Y_1))] = \int_{\mathbb{R}} \frac{\widetilde{\lambda}}{\lambda} \frac{d\widetilde{\nu}}{d\nu}(x)\, d\nu(x) = \frac{\widetilde{\lambda}}{\lambda} \int_{\mathbb{R}} d\widetilde{\nu}(x) = \frac{\widetilde{\lambda}}{\lambda}.$$

Moreover, for $a \in \mathbb{C}$ and $t \in [0,T]$,

$$E\left[a^{N_t}\right] = \sum_{k=0}^{\infty} \frac{(a\lambda t)^k}{k!} \exp(-\lambda t) = \exp((a-1)\lambda t). \tag{1}$$

Fix $t \in [0,T]$. Then by independence of N and the Y_k and by the above,

$$\begin{aligned} E[Z_t] &= \exp((\lambda - \widetilde{\lambda})t) E\left[E[\exp(\phi(Y_1)]^{N_t}\right] = \exp((\lambda - \widetilde{\lambda})t) E\left[(\widetilde{\lambda}/\lambda)^{N_t}\right] \\ &= \exp((\lambda - \widetilde{\lambda})t) \exp((\widetilde{\lambda}/\lambda - 1)\lambda t) = 1. \end{aligned}$$

(b) First, X has clearly RCLL paths under Q.

Next, we show that under Q, X has stationary increments and $X_t - X_s$ is independent of $\mathcal{F}_s$ for all $0 \le s < t \le T$. So fix $0 \le s < t \le T$ and let $g : \mathbb{R} \to \mathbb{R}$ be a bounded measurable function. Since X is a Lévy process for the filtration $(\mathcal{F}_t)_{t\in[0,T]}$ under P, it follows that the process $\widetilde{X} = (\widetilde{X}_u)_{u\in[0,T-s]}$ defined by $\widetilde{X}_u := X_{u+s} - X_s$ is independent of $\mathcal{F}_s$ and equal in distribution to $(X_u)_{u\in[0,T-s]}$ under P. Using this, the fact that

$$\begin{aligned} \frac{Z_t}{Z_s} &= \exp\left(\sum_{s<u\le t} \phi(\Delta X_u) + (\lambda - \widetilde{\lambda})(t-s)\right) \\ &= \exp\left(\sum_{0<u\le t-s} \phi(\Delta \widetilde{X}_u) + (\lambda - \widetilde{\lambda})(t-s)\right), \end{aligned}$$

the fact that Z is the density process of Q with respect to P on $(\mathcal{F}_t)_{t\in[0,T]}$ by part a) and Bayes' theorem gives

$$
\begin{aligned}
E_Q[g(X_t - X_s) \mid \mathcal{F}_s] &= E\left[\frac{Z_t}{Z_s} g(X_t - X_s) \,\middle|\, \mathcal{F}_s\right] \\
&= E\left[\left(\sum_{0<u\le t-s} \phi(\Delta \widetilde{X}_u) + (\lambda - \widetilde{\lambda})(t-s)\right) g(\widetilde{X}_{t-s})\right) \,\middle|\, \mathcal{F}_s\right] \\
&= E\left[\left(\sum_{0<u\le t-s} \phi(\Delta X_u) + (\lambda - \widetilde{\lambda})(t-s)\right) g(X_{t-s})\right)\right] \\
&= E[Z_{t-s} g(X_{t-s})] = E_Q[g(X_{t-s})].
\end{aligned}
$$

So X is a Lévy process for the filtration $(\mathcal{F}_t)_{t\in[0,T]}$ under Q. In order to show that it is even a compound Poisson process with rate $\widetilde{\lambda}$ and jump distribution $\widetilde{\nu}$, we calculate the characteristic function of X_1 under Q to determine its law. To this end, let $v \in \mathbb{R}$. First, note that

$$
E[\exp(ivY_1 + \phi(Y_1))] = \int_{\mathbb{R}} \exp(ivx) \frac{\widetilde{\lambda}}{\lambda} \frac{d\widetilde{\nu}}{d\nu}(x)\, d\nu(x) = \frac{\widetilde{\lambda}}{\lambda} \int_{\mathbb{R}} \exp(ivx)\, d\widetilde{\nu}(x).
$$

Using this, the independence of N and the Y_k under P and (1), gives

$$
\begin{aligned}
E_Q[\exp(ivX_1)] = E[Z_1 \exp(ivX_1)] &= \exp(\lambda - \widetilde{\lambda}) E\left[E[\exp(ivY_1 + \phi(Y_1))]^{N_1} \right] \\
&= \exp(\lambda - \widetilde{\lambda}) \exp\left(\left(\frac{\widetilde{\lambda}}{\lambda} \int_{\mathbb{R}} \exp(ivx)\, d\widetilde{\nu}(x) - 1 \right) \lambda \right) \\
&= \exp\left(\widetilde{\lambda} \int_{\mathbb{R}} (\exp(ivx) - 1)\, d\widetilde{\nu}(x) \right).
\end{aligned}
$$

Solution 7.2

(a) Set the process $R = (R_t)_{t\in[0,T]}$ by

$$
\begin{aligned}
R_t &:= \mu t + \frac{\sigma}{\sqrt{\lambda}} \widetilde{N}_t = \mu t + \frac{\sigma}{\sqrt{\lambda}}(N_t - \lambda t) = (\mu - \sigma\sqrt{\lambda})t + \frac{\sigma}{\sqrt{\lambda}} N_t \\
&= \frac{\sigma}{\sqrt{\lambda}}(N_t - \ell t), \quad t \in [0,T],
\end{aligned} \tag{2}
$$

where $\ell := \lambda - \frac{\mu}{\sigma}\sqrt{\lambda}$. Then $S = \mathcal{E}(R)$ and due to Exercise E.3 it is sufficient to find the EMM of R. The idea is to apply Exercise 7.1 with specified constants. Take $\tilde{\nu} = \nu$ and $\tilde{\lambda} = \ell$. It follows from the remark after Exercise E.3 that S fails NA, and a fortiori NFLVR, if the paths of R are monotone, i.e., if $\ell \le 0$. So we must have $\ell > 0$. Now define the measure $Q^\lambda \approx P$ on $\mathcal{F}_T$ by

$$
\frac{dQ^\lambda}{dP} = \exp\left(\sum_{k=1}^{N_T} \log \frac{\ell}{\lambda} + (\lambda - \ell)T \right). \tag{3}
$$

Then it follows from Exercise 7.1 that under Q^λ, $R_t = \frac{\sigma}{\sqrt{\lambda}}(N^{Q^\lambda} - \ell t), t \geq 0$, where $N^{Q^\lambda} := N$ is a Poisson process with rate ℓ. Since R is a Q^λ-martingale, it follows from Exercise E.3 that S is so, too.

(b) First observe that under Q^λ, $S = \mathcal{E}(R)$, where $R_t = \frac{\sigma}{\sqrt{\lambda}}(N^{Q^\lambda} - \ell t), t \geq 0$, with $N^{Q^\lambda} := N$ is a Poisson process with rate ℓ. Thus, using the given formula for $\mathcal{E}(R)$ and the facts that $\sum_{i=1}^T \Delta R_t = \sum_{i=1}^T \Delta N_t^{Q^\lambda} = N_T^{Q^\lambda}$, we obtain that

$$S_T = S_0 \exp\left(\log\left(1 + \frac{\sigma}{\sqrt{\lambda}}\right) N_T^{Q^\lambda} - \frac{\sigma\ell}{\sqrt{\lambda}} T\right).$$

Since S admits a unique equivalent martingale measure Q^λ (known from the hints), the risk-neutral price of $1_{\{S_T > K\}}$ is given by

$$\begin{aligned}
E_{Q^\lambda}[1_{\{S_T>K\}}] &= Q^\lambda[S_T > K] \\
&= Q^\lambda\left[S_0 \exp\left(\log\left(1 + \frac{\sigma}{\sqrt{\lambda}}\right) N_T^{Q^\lambda} - \frac{\sigma\ell}{\sqrt{\lambda}} T\right) > K\right] \\
&= Q^\lambda\left[N_T^{Q^\lambda} > \frac{\log\frac{K}{S_0} + \frac{\sigma\ell}{\sqrt{\lambda}} T}{\log\left(1 + \frac{\sigma}{\sqrt{\lambda}}\right)}\right] \\
&= \overline{\Psi}_{(\lambda - \frac{\mu}{\sigma}\sqrt{\lambda})T}\left(\frac{\log\frac{K}{S_0} + \left(\sigma\sqrt{\lambda} - \mu\right) T}{\log\left(1 + \frac{\sigma}{\sqrt{\lambda}}\right)}\right).
\end{aligned}$$

(c) First, define $\widetilde{Q}^\lambda \approx Q^\lambda$ on $\mathcal{F}_T$ by $\frac{d\widetilde{Q}^\lambda}{dQ^\lambda} := S_T/S_0$. Note that

$$S_T/S_0 = \mathcal{E}(R)_T = \exp\left(\sum_{k=1}^{N_T^{Q^\lambda}} \log\frac{\widetilde{\ell}}{\ell} + (\ell - \widetilde{\ell})T\right),$$

where $\widetilde{\ell} := \left(1 + \frac{\sigma}{\sqrt{\lambda}}\right)\ell$. Now it follows from Exercise 7.1 (b) that under $\widetilde{Q}^\lambda$,

$$R_t = \frac{\sigma}{\sqrt{\lambda}} N_t^{\widetilde{Q}^\lambda} - \frac{\sigma}{\sqrt{\lambda}} \ell t, \quad t \in [0, T],$$

where $N^{\widetilde{Q}^\lambda}$ is a Poisson process with rate $\widetilde{\ell}$.

Next, since S admits a unique equivalent martingale measure Q^λ (see hints), the arbitrage-free price of $S_T 1_{\{S_T>K\}}$ is given by $E_{Q^\lambda}[S_T 1_{\{S_T>K\}}]$. By Bayes' formula and the above and noting that under $\widetilde{Q}^\lambda$, the calculation is exactly the same as in part (a),

$$\begin{aligned}
E_{Q^\lambda}[S_T 1_{\{S_T>K\}}] &= E_{\widetilde{Q}^\lambda}[S_0 1_{\{S_T>K\}}] = S_0 \widetilde{Q}^\lambda[S_T > K] \\
&= S_0 \overline{\Psi}_{\left(1+\frac{\sigma}{\sqrt{\lambda}}\right)(\lambda - \frac{\mu}{\sigma}\sqrt{\lambda})T}\left(\frac{\log\frac{K}{S_0} + \left(\sigma\sqrt{\lambda} - \mu\right) T}{\log\left(1 + \frac{\sigma}{\sqrt{\lambda}}\right)}\right).
\end{aligned}$$

(d) First, it follows immediately from parts (a) and (b) that

$$\begin{aligned}
C_0^\lambda &= E_{Q^\lambda}[(S_T-K)^+] = E_{Q^\lambda}[S_T 1_{\{S_T>K\}}] - KE_{Q^\lambda}[1_{\{S_T>K\}}] \\
&= S_0\overline{\Psi}_{\left(1+\frac{\sigma}{\sqrt{\lambda}}\right)(\lambda-\frac{\mu}{\sigma}\sqrt{\lambda})T}\left(\frac{\log\frac{K}{S_0}+\left(\sigma\sqrt{\lambda}-\mu\right)T}{\log\left(1+\frac{\sigma}{\sqrt{\lambda}}\right)}\right) \\
&\quad - K\overline{\Psi}_{(\lambda-\frac{\mu}{\sigma}\sqrt{\lambda})T}\left(\frac{\log\frac{K}{S_0}+\left(\sigma\sqrt{\lambda}-\mu\right)T}{\log\left(1+\frac{\sigma}{\sqrt{\lambda}}\right)}\right).
\end{aligned}$$

Next, for $\rho>0$, let F_ρ be the distribution function of $\frac{X_\rho-\rho}{\sqrt{\rho}}$, where X_ρ is Poisson distributed with parameter ρ. Moreover, set $\overline{F}_\rho := 1-F_\rho$ and $\overline{\Phi}=1-\Phi$. Then by the hint, F_ρ converges pointwise to Φ as $\rho\to\infty$, and the convergence is even uniform as Φ is continuous. Thus $\overline{F}_\rho$ converges uniformly to $\overline{\Phi}$ as $\rho\to\infty$. Now the claim follows from the fact that $\overline{\Psi}_\rho(x)=\overline{F}_\rho\left(\frac{x-\rho}{\sqrt{\rho}}\right)$, the fact that $\overline{\Phi}(x)=\Phi(-x)$ and the limits

$$\begin{aligned}
&\lim_{\lambda\to\infty}\log\left(1+\frac{\sigma}{\sqrt{\lambda}}\right)\sqrt{\left(\lambda-\frac{\mu}{\sigma}\sqrt{\lambda}\right)T}=\sigma\sqrt{T}, \\
&\lim_{\lambda\to\infty}\log\left(1+\frac{\sigma}{\sqrt{\lambda}}\right)\sqrt{\left(1+\frac{\sigma}{\sqrt{\lambda}}\right)\left(\lambda-\frac{\mu}{\sigma}\sqrt{\lambda}\right)T}=\sigma\sqrt{T}, \\
&\lim_{\lambda\to\infty}\left(\left(\sigma\sqrt{\lambda}-\mu\right)T-\log\left(1+\frac{\sigma}{\sqrt{\lambda}}\right)\left(\lambda-\frac{\mu}{\sigma}\sqrt{\lambda}\right)T\right)=\frac{\sigma^2}{2}T, \\
&\lim_{\lambda\to\infty}\left(\left(\sigma\sqrt{\lambda}-\mu\right)T-\log\left(1+\frac{\sigma}{\sqrt{\lambda}}\right)\left(1+\frac{\sigma}{\sqrt{\lambda}}\right)\left(\lambda-\frac{\mu}{\sigma}\sqrt{\lambda}\right)T\right)=-\frac{\sigma^2}{2}T,
\end{aligned}$$

where we have used that

$$\begin{aligned}
\log\left(1+\frac{\sigma}{\sqrt{\lambda}}\right)&=\frac{\sigma}{\sqrt{\lambda}}-\frac{\sigma^2}{2\lambda}+O\left(\frac{1}{\lambda^{3/2}}\right), \\
\sqrt{\lambda-\frac{\mu}{\sigma}\sqrt{\lambda}}&=\sqrt{\lambda}\sqrt{1+O\left(\frac{1}{\sqrt{\lambda}}\right)}, \\
\sqrt{\left(1+\frac{\sigma}{\sqrt{\lambda}}\right)\left(\lambda-\frac{\mu}{\sigma}\sqrt{\lambda}\right)}&=\sqrt{\lambda}\sqrt{1+O\left(\frac{1}{\sqrt{\lambda}}\right)}.
\end{aligned}$$

Solution 7.3 Let Q be the martingale measure; we can write $\widetilde{S}_t=S_0\exp(\sigma W_t^Q+(r-\frac{1}{2}\sigma^2)t)$ for a Q-Brownian motion W^Q. Using the risk-neutral valuation formula, we have

$$\begin{aligned}
\widetilde{V}_t &= e^{-r(T-t)} E_Q[\widetilde{H}|\mathcal{F}_t] = e^{-r(T-t)} E_Q[1_{\{\widetilde{S}_T > \widetilde{K}\}}|\mathcal{F}_t] = e^{-r(T-t)} Q[\widetilde{S}_T > \widetilde{K}|\mathcal{F}_t] \\
&= e^{-r(T-t)} Q\left[\widetilde{S}_t \exp\Big(\sigma(W_T^Q - W_t^Q) + (r - \frac{1}{2}\sigma^2)(T-t)\Big) > \widetilde{K} \,\Big|\, \mathcal{F}_t\right] \\
&= e^{-r(T-t)} Q\left[-\sigma(W_T^Q - W_t^Q) < \ln \frac{x}{\widetilde{K}} + (r - \frac{1}{2}\sigma^2)(T-t)\right]\Bigg|_{x=\widetilde{S}_t} \\
&= e^{-r(T-t)} Q\left[\xi < \frac{\ln \frac{x}{\widetilde{K}} + (r - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}\right]\Bigg|_{x=\widetilde{S}_t} \\
&= e^{-r(T-t)} \Phi\left(\frac{\ln \frac{x}{\widetilde{K}} + (r - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}\right)\Bigg|_{x=\widetilde{S}_t} \\
&= \widetilde{v}(t, \widetilde{S}_t),
\end{aligned}$$

where $\xi = -(W_T^Q - W_t^Q)/\sqrt{T-t}$ has a standard Gaussian law and Φ is the standard normal c.d.f. As in the lecture, the strategy is given by the spatial derivative,

$$\widetilde{\vartheta}_t = \frac{\partial \widetilde{v}}{\partial x}(t, \widetilde{S}_t) = e^{-r(T-t)} \phi\left(\frac{\ln \frac{\widetilde{S}_t}{\widetilde{K}} + (r - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}\right) \frac{1}{\widetilde{S}_t \sigma \sqrt{T-t}}.$$

Here $\phi = \Phi'$ is the standard normal density.

Solution 7.4

(a) For an equivalent martingale measure Q, q_u, q_m and q_d must satisfy $q_m, q_d, q_u > 0$ (so that Q is equivalent to P) as well as

$$q_u + q_m + q_d = 1 \tag{4}$$

as well as $E^Q[S_1|\mathcal{F}_0] = S_0$, which, as $\mathcal{F}_0$ is trivial and $S_1 = S_0 Z$, can be written as the condition

$$q_d(1+d) + q_m(1+m) + q_u(1+u) = 1. \tag{5}$$

We see that for $d \geq 0$ and for $u \leq 0$, the system of equations (4) and (5) does not have any solution. This means that in these cases, there is no equivalent martingale measure which is by the fundamental theorem of asset pricing (FTAP) equivalent to the existence of arbitrage. For $u > 0 > d$, the system of equations (4) and (5) has solutions of the form

$$\begin{aligned}
q_d &:= \frac{-q_m(m-u) - u}{d-u} \in (0,1), \\
q_u &= 1 - q_m - \frac{-q_m(m-u) - u}{d-u} \in (0,1), \qquad q_m \in (0,1).
\end{aligned} \tag{6}$$

By the FTAP, this is equivalent to the absence of arbitrage.

(b) Assume that H is attainable. This means, as $\mathcal{F}_0$ is trivial, that there exist constants $x, \vartheta \in \mathbb{R}$ such that

$$x + \vartheta(S_1 - S_0) = x + \vartheta S_0(Z-1) = (S_1 - K)^+ = (S_0\, Z - K)^+. \tag{7}$$

On $\{Z = 1+u\}$, equation (7) can be read as

$$x + \vartheta S_0 u = S_0(1+u) - K. \tag{8}$$

On $\{Z = 1+m\}$, equation (7) can be read as

$$x + \vartheta S_0 m = S_0(1+m) - K. \tag{9}$$

Thus, subtracting (9) from (8) leads to $\vartheta = 1$ and $x = S_0 - K$. On the other hand, on $\{Z = 1+d\}$, equation (7) can be read as

$$x + \vartheta S_0 d = 0. \tag{10}$$

If $d = 0$, we obtain that $x = 0$, which implies that $0 = x = S_0 - K$. This contradicts the assumption that $S_0 \neq K$. If $d \neq 0$, we must have $\vartheta = \frac{-x}{S_0 d}$ for any initial value $x \in \mathbb{R}$. In particular, for $x = S_0 - K$, we obtain $\vartheta = \frac{-S_0+K}{S_0 d}$. Since we already have $\vartheta = 1$, we see that the equation

$$\frac{-S_0 + K}{S_0 d} = 1$$

must hold true. But this is only the case when $K = S_0(1+d)$, which was excluded in the assumption. Therefore $H = (S_1 - K)^+$ is not attainable.

Remark: We have not made any assumption about arbitrage in this part (b).

(c) *Case 1:* $\min_\Omega S_1 < K < \max_\Omega S_1$:
Denote $s_i^1 = S_1(\omega_i)$ for $i = d, m, u$. As $h(x) := (x-K)^+$ is convex, for any $Q' \in \mathbb{P}_e$, and for suitable $\lambda \in (0,1)$ such that $s_m^1 = \lambda s_d^1 + (1-\lambda)s_u^1$ (or equivalently, such that $1+m = \lambda(1+d) + (1-\lambda)(1+u)$, hence $\lambda = \frac{u-m}{u-d}$),

$$\begin{aligned} E_{Q'}[h(S_1)] &= q'_d h(s_d^1) + q'_m h(\lambda s_d^1 + (1-\lambda)s_u^1) + q'_u h(s_u^1) \\ &\leq q'_d h(s_d^1) + q'_m \lambda h(s_d^1) + q'_m(1-\lambda)h(s_u^1) + q'_u h(s_u^1) \\ &= [q'_d + \lambda q'_m]h(s_d^1) + [q'_u + (1-\lambda)q'_m]h(s_u^1) =: q_d h(s_d^1) + q_u h(s_u^1) = E_Q[h(S_1)] \end{aligned}$$

for $Q := (q_d, 0, q_u)$. We point out, using (6), that we have $q_d = q'_d + \lambda q'_m = \frac{u}{u-d}$ and that $q_u = q'_u + (1-\lambda)q'_m = \frac{-d}{u-d}$, which is independent of the choice of $Q' \in \mathbb{P}_e$, hence Q is well-defined. Clearly, Q is a probability measure as $q_d, q_u > 0$ and $q_d + q_u = 1$. Moreover, $Q \ll P$, but Q is not equivalent to P as Q does not have any mass in ω_m. To be a martingale measure, we need to have that $E_Q[Z] = 1$ or equivalently, that

$$q_d(1+d) + q_u(1+u) = 1,$$

which is satisfied. Now, we know from above that $\pi^s(H) := \sup_{Q' \in \mathbb{P}_e} E_{Q'}[H] \le E_Q[H]$. We claim that $\sup_{Q' \in \mathbb{P}_e} E_{Q'}[H] = E_Q[H]$. For that purpose, consider for each n a measure Q^n with corresponding $q_d^n, q_m^n, q_u^n > 0$ satisfying (6) such that $(q_m^n)_{n\in\mathbb{N}}$ converges to 0. Clearly, from (a), we have $(Q^n) \subseteq \mathbb{P}_e$. Moreover, from the above calculation, we see that $\lim_{n\to\infty} E^{Q^n}[H] = E^Q[H]$. Thus, we see that

$$\sup_{Q' \in \mathbb{P}_e} E_{Q'}[H] \ge \lim_{n\to\infty} E_{Q^n}[H] = E_Q[H],$$

which implies that $\sup_{Q' \in \mathbb{P}_e} E_{Q'}[H] = E_Q[H]$. Thus, since in this case $H(\omega_d) = 0$, we get that

$$\pi^s(H) = E_Q[H] = q_u(S_0(1+u) - K) + 0 = \frac{-d}{u-d}\big(S_0(1+u) - K\big).$$

Case 2: $K > \max_\Omega S_1$: Then $H = 0$ and thus $\pi_s(H) = 0$.
Case 3: $K < \min_\Omega S_1$: Then $H = S_1 - K$ is attainable with admissible strategy $(V_0, \vartheta) = (S_0 - K, 1)$. In particular $\pi_s(H) = V_0 = S_0 - K$.
Obviously in Case 2 and Case 3, any martingale measure Q which is absolutely continuous to P satisfies that $\pi_s(H) = E_Q[H]$.

Solution 7.5

(a) i) is correct. This is a part of the proof of Theorem 6.3 of the lecture. More precisely, if S does not satisfy the NA property, then of course $\#(\mathbb{P}_e) = 0$ due to the DMW theorem. If S satisfies NA, then $\mathbb{P}_e \ne \emptyset$. Let us assume Q_1 and Q_2 are two elements in $\mathbb{P}_e$. Since $(S, \mathbb{F})$ is complete, for any bounded and $\mathcal{F}$-measurable random variable H, there is an integrand ϑ such that $H = H_0 + \vartheta \bullet S_T$, where H_0 is a constant (since $\mathcal{F}_0$ is trivial). It follows then that

$$E_{Q_1}[H] = H_0 + E_{Q_1}[\vartheta \bullet S_T] = H_0 = H_0 + E_{Q_2}[\vartheta_s \bullet S_T] = E_{Q_2}[H],$$

because in finite discrete time, ϑ admissible implies that $\vartheta \bullet S$ is Q-martingale for any ELMM Q for S. Clearly, by taking $H = 1_A$ for any $A \in \mathcal{F}$ we can derive that $Q_1 = Q_2$.

ii) is wrong. Take the market as in Exercise 7.5 with $d > 0$. Then, from Exercise 7.5 (a), we know that there is arbitrage, thus $\mathbb{P}_e = \emptyset$ by the fundamental theorem of asset pricing. But in Exercise 7.5 (b), we find an option H not being attainable.

(b) Suppose that NA holds under P. Let $\tilde{R}$ be the measure

$$\frac{\mathrm{d}\tilde{R}}{\mathrm{d}P} = \frac{E[(1+H)^{-1}]}{1+H}$$

so that $\tilde{R} \approx P$ with

$$E_R[H] = E\left[\frac{HE[(1+H)^{-1}]}{1+H}\right] < \infty.$$

Therefore we still have NA under $\tilde{R}$, and by Dalang-Morton-Willinger, we can obtain an EMM $R \in \mathbb{P}_e(S)$ with bounded $\xi := d\tilde{R}/dR$. Clearly, $E_R[H] = E_{\tilde{R}}[\xi H] \le \|\xi\|_\infty E_R[H] < \infty$.

Solution 7.6

(a) Let $(\Omega^1, \mathcal{F}^1, P^1)$ be a probability space with a Brownian motion W^1 and its natural filtration $\mathbb{F}^1 := \mathbb{F}^{W^1}$. Let $(\Omega', \mathcal{G}, R)$ be another probability space carrying the random variables b', σ' with distributions as desired for b, σ. Elements of Ω' are denoted ω'. Let $(\Omega, \mathbb{F}, P)$ be the product of the two spaces, with the filtration $\mathbb{F} = (\mathcal{F}_t)$, $\mathcal{F}_t := \mathcal{F}^1_t \otimes \mathcal{G}$. In this space, the elements are labelled $\omega = (\omega^1, \omega')$. Define the random variables $b(\omega^1, \omega') := b'(\omega')$, $\sigma(\omega^1, \omega') := \sigma'(\omega')$, and $W(\omega^1, \omega') = W^1(\omega^1)$. Then we have a model on Ω as required.

(b) Note that $\mathcal{F}_0$ can be identified with $\mathcal{G}$ a.s. Given $\mathcal{G}$ (and hence b, σ), the model is a Black-Scholes with unique EMM density process $\mathcal{E}(-\lambda(\omega') \bullet W)$, where $\lambda := b/\sigma$. The only additional degree of freedom is therefore a choice at time zero. We need to have a positive martingale with expectation 1, so the most general expression is $Z = \xi(\omega')\mathcal{E}(-\lambda(\omega') \bullet W)$, where ξ is $\mathcal{F}_0$ measurable (and hence independent of W), $\xi > 0$, and $E[\xi] = 1$. Conversely, given $\mathcal{F}_0$, such a Z is clearly a martingale, and therefore it is also martingale unconditionally. The same argument applies to ZS, as Z and S given $\mathcal{F}_0$ describe the Black-Scholes EMM density (up to a constant) and the Black-Scholes price process, respectively.

(c) For $Q \in \mathbb{P}_{\mathrm{e,loc}}$, we recall from part (b) that the density process Z^Q is of the form $\xi(\omega')\mathcal{E}(-\lambda(\omega') \bullet W)$, where $\lambda = b/\sigma$, ξ is $\mathcal{F}_0$-measurable and hence independent of W, $\xi > 0$, and $E[\xi] = 1$. Using that $dQ/dP|_{\mathcal{F}_0} = \xi$, we obtain

$$\begin{aligned} E_Q[H] &= E[Z^Q_T H] = E[E\{Z^Q_T H|\mathcal{F}_0\}] = E[\xi E\{\mathcal{E}(-\lambda(\omega') \bullet W)_T H|\mathcal{F}_0\}] \\ &= E[\xi\, u_{BS}(0, S_0, \sigma)] = E_Q[u_{BS}(0, S_0, \sigma)] = \int_0^\infty u_{BS}(0, S_0, v)\, \nu^Q_\sigma(dv). \end{aligned}$$

(d) We first calculate $\lim_{v\to\pm\infty} u_{BS}(0, S_0, v)$. Clearly $d_\pm \to \pm\infty$ as $v \to \infty$ and $d_\pm \to \operatorname{sign}(\log(S_0/K)) \times \infty$ as $v \to 0$ (where $\operatorname{sign}(0) := 0$ and $0 \cdot \infty = 0$). Hence, using $\Phi(-\infty) = 0$, $\Phi(0) = 1/2$, $\Phi(+\infty) = 1$,

$$\lim_{v\to\infty} u_{BS}(0, S_0, v) = S_0$$

and

$$\lim_{v\to 0} u_{BS}(0, S_0, v) = \left\{ \begin{array}{rl} S_0 - K & \text{if } S_0 > K \\ 0 & \text{if } S_0 < K \\ (S_0 - K)/2 = 0 & \text{if } S_0 = K \end{array} \right\} = (S_0 - K)^+.$$

Note also that $u_{BS}(0, S_0, \cdot)$ is uniformly bounded. Now recall again the formula from (b) or a general EMM density, which leads us to

$$\pi_s(H) = \sup_Q E_Q[H] = \sup_\xi E[\xi\, u_{BS}(0, S_0, \sigma)],$$

where ξ ranges over $\Xi = \{\xi \in L^1_{++}(\mathcal{F}_0),\ E[\xi] = 1\}$. We concentrate Q where σ is large: Let $A_n = \{\sigma \geq n\}$ and $\xi_n = c_n[(1 - 1/n)1_{A_n} + (1/n)1_{A_n^c}]$, where c_n is such that $E[\xi_n] = 1$. Then $\xi_n \in \Xi$ and thus

$$\pi_s(H) \geq \limsup_n E[\xi_n\, u_{BS}(0, S_0, \sigma)] = u_{BS}(0, S_0, \infty) = S_0.$$

The converse inequality is clear because one can superreplicate the call by just buying the stock with S_0. Similarly,

$$\pi_b(H) = \inf_Q E_Q[H] \leq \liminf_n E[\xi'_n\, u_{BS}(0, S_0, \sigma)] = u_{BS}(0, S_0, 0) = (S_0 - K)^+$$

by choice of suitable ξ'_n concentrating on $\{\sigma < 1/n\}$. Again, the converse inequality is clear.

The result means that the optimal way of superreplicating the call is to simply buy the stock at time zero, and the optimal way of subreplicating it is either to do nothing at all (if $S_0 \leq K$) or to short sell $S - K$ (if $S_0 > K$).

This means that π_s, π_b are too extreme to be of practical use for pricing in this market, and in fact in most real markets.