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Mathematical Finance

Solution sheet 8

Solution 8.1

(a) i) If $x \in \mathbb{R}$ and $\vartheta \in \Theta_{\mathrm{adm}}$ with $x+\vartheta \bullet S_T \geq H'$, then, as $H \leq H'$, $x+\vartheta \bullet S_T \geq H$. Thus, by definition $\pi^s(H) \leq \pi^s(H')$.

ii) For $y := x - c$ we obtain that

$$\pi^s(H+c) = \inf\{x : x+\vartheta \bullet S_T \geq H+c\} = \inf\{y : y+\vartheta \bullet S_T \geq H\}+c = \pi^s(H)+c.$$

iii) If $x \geq \pi^s(H)$, then there is a $\vartheta \in \Theta_{\mathrm{adm}}$ such that $x + \vartheta \bullet S_T \geq H$. This implies that $\lambda x + \lambda\vartheta \bullet S_T \geq \lambda H$, which gives us $\lambda\pi^s(H) \geq \pi^s(\lambda H)$. For the other inequality, just note that the same argument applied for λH and $1/\lambda$ yields that $\frac{1}{\lambda}\pi^s(\lambda H) \geq \pi^s(H)$.

(b) i) S satisfies (NA) $\iff$ $\mathcal{G}_{\mathrm{adm}} \cap L^0_+ = \{0\}$ $\iff$ $\forall g \in \mathcal{G}_{\mathrm{adm}}$, $g \geq 0 \Rightarrow g = 0$ $\iff$ 0 is an maximal element in $\mathcal{G}_{\mathrm{adm}}$.

ii) If $S = (S_k)_{k=0,1,\ldots,T}$ satisfies (NA), then by Theorem 1.2 in the lecture notes, we have $\mathcal{G}_T(\Theta) \cap L^0_+ = \{0\}$. If $G_T(\vartheta) \in G_T(\Theta)$ is not a maximal element, then we can find some $\psi \in \Theta$ such that $G_T(\psi) \geq G_T(\vartheta)$ and $P[G_T(\psi) > G_T(\vartheta)] > 0$. This means that $G_T(\psi - \vartheta) \geq 0$ and $P[G_T(\psi - \vartheta) > 0] > 0$. But since $\psi - \vartheta$ is an element in Θ, this contradicts the fact that $\mathcal{G}_T(\Theta) \cap L^0_+ = \{0\}$. Hence every element in $G_T(\Theta)$ is maximal in $G_T(\Theta)$. Clearly, since $\mathcal{G}_{\mathrm{adm}} \subset G_T(\Theta)$, the same conclusion holds also for $\mathcal{G}_{\mathrm{adm}}$.
We can also prove this result for $\mathcal{G}_{\mathrm{adm}}$ directly: if $S = (S_k)_{k=0,1,\ldots,T}$ satisfies (NA), then by Corollary 1.3 in the lecture notes, there is an equivalent martingale measure Q under which S is a discrete-time martingale. Then, for all $\vartheta \in \Theta_{\mathrm{adm}}$, $\vartheta \bullet S$ is also a Q-martingale.

Warning: this fact only holds for *finite discrete-time models*, for continuous-time models $\vartheta \bullet S$ is not a martingale in general! Hence we have $E_Q[g] = 0$ for all $g \in \mathcal{G}_{\mathrm{adm}}$, which of course implies that g is maximal in $\mathcal{G}_{\mathrm{adm}}$. (Indeed, if $h \geq g$, $h > g$ with positive probability for some $h \in \mathcal{G}_{\mathrm{adm}}$, then $E_Q[h] > 0$, which is a contradiction to $E_Q[h] = 0$.

Remark: Every *discrete-time* local martingale bounded from below is a true martingale.

Solution 8.2

(a) We first show that u is increasing. Let $x, y \in (0,\infty)$ with $x \le y$ and $\vartheta \in \Theta^x_{\mathrm{adm}}$. Clearly $x + \vartheta \bullet S_T \le y + \vartheta \bullet S_T$. Because $\Theta^x_{\mathrm{adm}} \subseteq \Theta^y_{\mathrm{adm}}$ and U is increasing, we have

$$E[U(V_T(\vartheta))] \le E[U(y + \vartheta \bullet S_T)] \le u(y).$$

Taking the supremum over $\mathcal{V}(x)$ on the LHS yields $u(x) \le u(y)$.

Now we prove the concavity of u. Let $\lambda \in [0,1]$ and $x, y \in (0,\infty)$ with $x \le y$. If $\vartheta^x \in \Theta^x_{\mathrm{adm}}$ and $\vartheta^y \in \Theta^y_{\mathrm{adm}}$, we clearly have $\lambda\vartheta^x + (1-\lambda)\vartheta^y \in \Theta^{\lambda x+(1-\lambda)y}_{\mathrm{adm}}$. So

$$\begin{aligned} u\Big(\lambda(x) + (1-\lambda)y\Big) &\ge E\Big[U\Big(\lambda(x + \vartheta^x \bullet S_T) + (1-\lambda)(y + \vartheta^y \bullet S_T)\Big)\Big] \\ &\ge \lambda E[U(x + \vartheta^x \bullet S_T)] + (1-\lambda)E[U(y + \vartheta^y \bullet S_T)]. \end{aligned}$$

Taking the supremum over $\mathcal{V}(x)$ and $\mathcal{V}(y)$ on the RHS yields $u(\lambda(x) + (1-\lambda)y) \ge \lambda u(x) + (1-\lambda)u(y)$.

(b) By part (a), we only need to prove $u(x) < \infty$ for all $x \in (x_0, \infty)$. But clearly we can find $\lambda \in (0,1)$ and $x < y$ such that $x_0 = \lambda x + (1-\lambda)y$. So by the concavity of u, we have $\lambda u(x) + (1-\lambda)u(y) \le u(x_0)$ which implies

$$u(x) \le \frac{u(x_0) - (1-\lambda)u(y)}{\lambda} < \infty.$$

(c) Suppose to the contrary that we have $u(x) \ge U(\infty)$ for some $x \in (0,\infty)$. It is clear that $U(V_T) \le U(\infty)$ for all $V \in \mathcal{V}(x)$ and hence $u(x) = \sup_{V\in\mathcal{V}(x)} E[U(V_T)] \le U(\infty)$. So we must have $u(x) = U(\infty)$. Let $(V^n) \subseteq \mathcal{V}(x)$ be such that $E[U(V^n_T)] \uparrow U(\infty)$. By Lemma 4.4, for each $n \in \mathbb{N}$, there exists $\tilde{V}^n_T \in \mathrm{conv}(V^n_T, V^{n+1}_T, \dots)$ such that $\tilde{V}^n_T \to V^\infty$ P-a.s.. The assumption NFLVR, in particular NUPBR, implies that $\mathrm{conv}(V^1_T, V^2_T, \dots)$ is bounded in L^0 and hence by Lemma 4.4 again, $\tilde{V}^\infty < \infty$ P-a.s. The concavity of U implies that $E[U(\tilde{V}^n_T)] \ge \inf_{k\ge n} E[U(V^k_T)] = E[U(V^n_T)]$. Since $U(\tilde{V}^n_T) \le U(\infty)$ for all $n \in \mathbb{N}$, applying the reverse Fatou lemma gives

$$E[U(V^\infty)] \ge \limsup_{n\to\infty} E[U(\tilde{V}^n_T)] \ge \liminf_{n\to\infty} E[U(V^n_T)] = U(\infty).$$

So clearly $E[U(\infty) - U(V^\infty)] = 0$. But U is strictly increasing and $V^\infty < \infty$ P-a.s., so $U(\infty) - U(\tilde{V}^\infty_T) > 0$ P-a.s. which gives a contradiction.

Solution 8.3

(a) Let Z be the density process of Q w.r.t. P. Suppose there exists $h \in \mathcal{D}(z)$ with $A := \{h > zZ_T\}$ having $P[A] > 0$ for. Define $M_t := E_Q[\mathbb{1}_A|\mathcal{F}_t]$. Then $M \ge 0$ and $M_0 = Q[A] > 0$ due to $Q \approx P$. Clearly $E_Q[M_T] \le M_0$ for all Q. By Lemma 6.2, This implies $M_T \in \mathcal{V}(M_0)$ and so $E[hM_T] \le zM_0$ by definition of $\mathcal{D}(z)$. On the other hand, $E[zZ_TM_T] = E_Q[zM_T] = zM_0$. It follows $E[(h - zZ_T)M_T] \le 0$. But clearly $E[(h-zZ_T)M_T] = E[(h-zZ_T)\mathbb{1}_A] > 0$ which gives a contradiction. The other claim easily follows from the first claim.

(b) The process S^1 satisfies

$$\mathrm{d}S^1_t = S^1_t\Big((\mu - r)\,\mathrm{d}t + \sigma\,\mathrm{d}W_t\Big).$$

Also recall that S has a unique EMM Q on $\mathcal{F}_T$ which has density

$$\frac{\mathrm{d}Q}{\mathrm{d}P} = \mathcal{E}(-\lambda W)_T,$$

where $\lambda := (\mu - r)/\sigma$. It is also easy to calculate

$$J(z) = \frac{1-\gamma}{\gamma} z^{-\frac{-\gamma}{1-\gamma}} \text{ and } J'(z) = -z^{-\frac{1}{1-\gamma}}.$$

Then by part (a) and the fact that $\mathcal{E}(aW)$ is a P-martingale for every $a \in \mathbb{R}$,

$$\begin{aligned}
j(z) &= E\left[\frac{1-\gamma}{\gamma} z^{-\frac{\gamma}{1-\gamma}} \Big(\mathcal{E}(-\lambda W)_T\Big)^{-\frac{\gamma}{1-\gamma}}\right] \\
&= \frac{1-\gamma}{\gamma} z^{-\frac{\gamma}{1-\gamma}} E\left[\exp\left(\frac{\lambda\gamma}{1-\gamma} W_T + \frac{1}{2}\frac{\lambda^2\gamma}{1-\gamma} T\right)\right] \\
&= \frac{1-\gamma}{\gamma} z^{-\frac{\gamma}{1-\gamma}} \exp\left(\frac{1}{2}\frac{\lambda^2\gamma}{(1-\gamma)^2} T\right) E\left[\mathcal{E}\left(\frac{\lambda\gamma}{1-\gamma} W\right)_T\right] \\
&= \frac{1-\gamma}{\gamma} z^{-\frac{\gamma}{1-\gamma}} \exp\left(\frac{1}{2}\frac{\lambda^2\gamma}{(1-\gamma)^2} T\right) < \infty.
\end{aligned}$$

(c) "$\le$" is clear. For "$\ge$", if we justify the hint, then $J(h) \ge J(z\frac{\mathrm{d}Q}{\mathrm{d}P})$ and $E[J(h)] \ge$ RHS for all $h \in \mathcal{D}(z)$. Let $U = \operatorname{ess\,sup}_{Q \in \mathbb{P}_{e,\sigma}} Z^{Q;P}$. Suppose to the contrary that $A := \{h > zU_T\}$ has $P[A] > 0$. Then for some $\delta > 0$, the event $A_\delta := \{h - zU_T \ge \delta\} \subset A$ has $P[A_\delta] > 0$. Let $\varepsilon > 0$ be arbitrary. Choose $\tilde{Q} \in \mathbb{P}_{e,\sigma}$ with $\tilde{Q}[A] \ge \sup_Q Q[A] - \varepsilon$ and define $M_t := E_{\tilde{Q}}[\mathbb{1}_A|\mathcal{F}_t]$. Then $\sup_Q E_Q[M_T] = \sup_Q E_Q[A] \le \tilde{Q}[A] + \varepsilon = M_0 + \varepsilon$. By Lemma 10.1, this implies $M_T \in \mathcal{C}(M_0 + \varepsilon)$ and so by definition $M_T \le V_T$ for some $V_T \in \mathcal{V}(M_0 + \varepsilon)$. Therefore $E[hM_T] \le z(M_0 + \varepsilon)$ by definition of $\mathcal{D}(z)$. However, $E[zZ^{\tilde{Q};P}_T M_T] = E_{\tilde{Q}}[zM_T] = zM_0$. It follows $E[(h - zZ^{\tilde{Q};P})M_T] \le z\varepsilon$. But clearly

$$E[(h - z\underbrace{Z^{\tilde{Q};P}_T}_{\le U_T})M_T] \ge E[(h - zU_T)\mathbb{1}_A] \ge E[(h - zU_T)\mathbb{1}_{A_\delta}] \ge \delta P[A_\delta].$$

Hence $\delta P[A_\delta] \le z\varepsilon$ but sending $\varepsilon \to 0$ implies $P[A_\delta] = 0$. This is a contradiction.