Mathematical Finance
Solution sheet 8

Solution 8.1

(a) i) If \( x \in \mathbb{R} \) and \( \vartheta \in \Theta_{\text{adm}} \) with \( x + \vartheta \cdot S_T \geq H' \), then, as \( H \leq H' \), \( x + \vartheta \cdot S_T \geq H \). Thus, by definition \( \pi^s(H) \leq \pi^s(H') \).

ii) For \( y := x - c \) we obtain that
\[
\pi^s(H + c) = \inf \{ x : x + \vartheta \cdot S_T \geq H + c \} = \inf \{ y : y + \vartheta \cdot S_T \geq H \} + c = \pi^s(H) + c.
\]

iii) If \( x \geq \pi^s(H) \), then there is a \( \vartheta \in \Theta_{\text{adm}} \) such that \( x + \vartheta \cdot S_T \geq H \). This implies that \( \lambda x + \lambda \vartheta \cdot S_T \geq \lambda H \), which gives us \( \lambda \pi^s(H) \geq \pi^s(\lambda H) \). For the other inequality, just note that the same argument applied for \( \lambda H \) and \( \lambda^{-1} \) yields that \( \lambda^{-1} \pi^s(\lambda H) \geq \pi^s(H) \).

(b) i) \( S \) satisfies (NA) \( \iff \) \( G_{\text{adm}} \cap L^0_+ = \{0\} \iff \forall g \in G_{\text{adm}}, g \geq 0 \Rightarrow g = 0 \iff 0 \) is a maximal element in \( G_{\text{adm}} \).

ii) If \( S = (S_k)_{k=0,1,...,T} \) satisfies (NA), then by Theorem 1.2 in the lecture notes, we have \( G_T(\Theta) \cap L^0_+ = \{0\} \). If \( G_T(\vartheta) \in G_T(\Theta) \) is not a maximal element, then we can find some \( \psi \in \Theta \) such that \( G_T(\psi) \geq G_T(\vartheta) \) and \( P[G_T(\psi) > G_T(\vartheta)] > 0 \). This means that \( G_T(\psi - \vartheta) \geq 0 \) and \( P[G_T(\psi - \vartheta) > 0] > 0 \). But since \( \psi - \vartheta \) is an element in \( \Theta \), this contradicts the fact that \( G_T(\Theta) \cap L^0_+ = \{0\} \). Hence every element in \( G_T(\Theta) \) is maximal in \( G_T(\Theta) \). Clearly, since \( G_{\text{adm}} \subset G_T(\Theta) \), the same conclusion holds also for \( G_{\text{adm}} \).

We can also prove this result for \( G_{\text{adm}} \) directly: if \( S = (S_k)_{k=0,1,...,T} \) satisfies (NA), then by Corollary 1.3 in the lecture notes, there is an equivalent martingale measure \( Q \) under which \( S \) is a discrete-time martingale. Then, for all \( \vartheta \in \Theta_{\text{adm}}, \vartheta \cdot S \) is also a \( Q \)-martingale.

**Warning:** this fact only holds for **finite discrete-time models**, for continuous-time models \( \vartheta \cdot S \) is not a martingale in general! Hence we have \( E_Q[g] = 0 \) for all \( g \in G_{\text{adm}} \), which of course implies that \( g \) is maximal in \( G_{\text{adm}} \). (Indeed, if \( h \geq g \), \( h > g \) with positive probability for some \( h \in G_{\text{adm}} \), then \( E_Q[h] > 0 \), which is a contradiction to \( E_Q[g] = 0 \).)

**Remark:** Every **discrete-time** local martingale bounded from below is a true martingale.

Solution 8.2

Updated: December 14, 2018
(a) We first show that \( u \) is increasing. Let \( x, y \in (0, \infty) \) with \( x \leq y \) and \( \vartheta \in \Theta_{adm}^x \). Clearly \( x + \vartheta \cdot S_T \leq y + \vartheta \cdot S_T \). Because \( \Theta_{adm}^x \subseteq \Theta_{adm}^y \) and \( U \) is increasing, we have

\[
E[U(V_T(\vartheta))] = E[U(y + \vartheta \cdot S_T)] \leq u(y).
\]

Taking the supremum over \( \mathcal{V}(x) \) on the LHS yields \( u(x) \leq u(y) \).

Now we prove the concavity of \( u \). Let \( \lambda \in [0, 1] \) and \( x, y \in (0, \infty) \) with \( x \leq y \). If \( \vartheta^x \in \Theta_{adm}^x \) and \( \vartheta^y \in \Theta_{adm}^y \), we clearly have \( \lambda \vartheta^x + (1 - \lambda) \vartheta^y \in \Theta_{adm}^{\lambda x + (1 - \lambda) y} \). So

\[
u\left(\lambda(x) + (1 - \lambda)y\right) \geq E\left[U\left(\lambda(x + \vartheta^x \cdot S_T) + (1 - \lambda)(y + \vartheta^y \cdot S_T)\right)\right]
\geq \lambda E[U(x + \vartheta^x \cdot S_T)] + (1 - \lambda)E[U(y + \vartheta^y \cdot S_T)].
\]

Taking the supremum over \( \mathcal{V}(x) \) and \( \mathcal{V}(y) \) on the RHS yields

\[
u(\lambda(x) + (1 - \lambda)y) \geq \lambda u(x) + (1 - \lambda)u(y).
\]

(b) By part (a), we only need to prove \( u(x) \leq \infty \) for all \( x \in (0, \infty) \). But clearly we can find \( \lambda \in (0, 1) \) and \( x < y \) such that \( x_0 = \lambda x + (1 - \lambda)y \). So by the concavity of \( u \), we have \( \lambda u(x) + (1 - \lambda) u(y) \leq u(x_0) \) which implies

\[
u(x) \leq \frac{u(x_0) - (1 - \lambda)u(y)}{\lambda} < \infty.
\]

(c) Suppose to the contrary that we have \( u(x) \geq U(\infty) \) for some \( x \in (0, \infty) \). It is clear that \( U(V_T^x) \leq U(\infty) \) for all \( V \in \mathcal{V}(x) \) and hence \( u(x) = \sup_{V \in \mathcal{V}(x)} E[U(V_T^x)] \leq U(\infty) \). So we must have \( u(x) = U(\infty) \). Let \( (V_n) \subseteq \mathcal{V}(x) \) be such that \( E[U(V_T^n)] \uparrow U(\infty) \). By Lemma 4.4, for each \( n \in \mathbb{N} \), there exists \( \tilde{V}_T^n \in \text{conv}(V_T^n, V_T^{n+1}, \ldots) \) such that \( \tilde{V}_T^n \rightarrow V^\infty \) P-a.s.. The assumption NFLVR, in particular NUPBR, implies that \( \text{conv}(V_T^1, V_T^2, \ldots) \) is bounded in \( L^0 \) and hence by Lemma 4.4 again, \( V^\infty < \infty \) P-a.s. The concavity of \( U \) implies that \( E[U(V_T^\infty)] \geq \inf_{k \geq n} E[U(V_T^k)] = E[U(V_T^n)] \). Since \( U(\tilde{V}_T^n) \leq U(\infty) \) for all \( n \in \mathbb{N} \), applying the reverse Fatou lemma gives

\[
E[U(V^\infty)] \leq \limsup_{n \to \infty} E[U(\tilde{V}_T^n)] \geq \liminf_{n \to \infty} E[U(V_T^n)] = U(\infty).
\]

So clearly \( E[U(\infty) - U(V^\infty)] = 0 \). But \( U \) is strictly increasing and \( V^\infty < \infty \) P-a.s., so \( U(\infty) - U(\tilde{V}_T^\infty) > 0 \) P-a.s. which gives a contradiction.

Solution 8.3

(a) Let \( Z \) be the density process of \( Q \) w.r.t. \( P \). Suppose there exists \( h \in \mathcal{D}(z) \) with \( A := \{ h > zT_T \} \) having \( P[A] > 0 \) for. Define \( M_t := \mathbb{E}_Q[I_A|\mathcal{F}_t] \). Then \( M \geq 0 \) and \( M_0 = Q[A] > 0 \) due to \( Q \approx P \). Clearly \( E_Q[M_T] \leq M_0 \) for all \( Q \). By Lemma 6.2, this implies \( M_T \in \mathcal{V}(M_0) \) and so \( E[hM_T] \leq zM_0 \) by definition of \( \mathcal{D}(z) \). On the other hand, \( E[zZ_TM_T] = E_Q[zM_T] = zM_0 \). It follows \( E[(h - zZ_T)M_T] \leq 0 \). But clearly \( E[(h - zZ_T)M_T] = E[(h - zZ_T)I_A] > 0 \) which gives a contradiction. The other claim easily follows from the first claim.
(b) The process $S^1$ satisfies
\[ dS^1_t = S^1_t \left( (\mu - r) \, dt + \sigma \, dW_t \right). \]

Also recall that $S$ has a unique EMM $Q$ on $\mathcal{F}_T$ which has density
\[ \frac{dQ}{dP} = \mathcal{E}(-\lambda W)_T, \]
where $\lambda := (\mu - r)/\sigma$. It is also easy to calculate
\[ J(z) = \frac{1 - \gamma}{\gamma} z^{-\frac{\gamma}{1 - \gamma}} \ \text{and} \ J'(z) = -z^{-\frac{1}{1 - \gamma}}. \]

Then by part (a) and the fact that $\mathcal{E}(aW)$ is a $P$-martingale for every $a \in \mathbb{R}$,
\[ j(z) = E \left[ \frac{1 - \gamma}{\gamma} z^{-\frac{\gamma}{1 - \gamma}} \left( \mathcal{E}(-\lambda W)_T \right)^{-\frac{\gamma}{1 - \gamma}} \right] \]
\[ = \frac{1 - \gamma}{\gamma} z^{-\frac{\gamma}{1 - \gamma}} E \left[ \exp \left( \frac{\lambda \gamma}{1 - \gamma} W_T + \frac{1}{2} \frac{\lambda^2 \gamma}{1 - \gamma} T \right) \right] \]
\[ = \frac{1 - \gamma}{\gamma} z^{-\frac{\gamma}{1 - \gamma}} \exp \left( \frac{1}{2} \frac{\lambda^2 \gamma}{(1 - \gamma)^2} T \right) E \left[ \mathcal{E} \left( \frac{\lambda \gamma}{1 - \gamma} W \right) \right] \]
\[ = \frac{1 - \gamma}{\gamma} z^{-\frac{\gamma}{1 - \gamma}} \exp \left( \frac{1}{2} \frac{\lambda^2 \gamma}{(1 - \gamma)^2} T \right) < \infty. \]

(c) “$\leq$” is clear. For “$\geq$”, if we justify the hint, then $J(h) \geq J(z \frac{dQ}{dP})$ and $E[J(h)] \geq \text{RHS}$ for all $h \in \mathcal{D}(z)$. Let $U = \text{ess sup}_{Q \in \mathcal{P}_{e,\sigma}} Z_{Q-P}$. Suppose to the contrary that $A := \{ h > zU_T \}$ has $P[A] > 0$. Then for some $\delta > 0$, the event $A_\delta := \{ h - zU_T \geq \delta \} \subset A$ has $P[A_\delta] > 0$. Let $\varepsilon > 0$ be arbitrary. Choose $\bar{Q} \in \mathcal{P}_{e,\sigma}$ with $\bar{Q}[A] \geq \sup_Q Q[A] - \varepsilon$ and define $M_t := \bar{E}_\bar{Q}[1_A | \mathcal{F}_t]$. Then $\sup_Q E_{\bar{Q}}[M_T] = \sup_Q E_Q[A] \leq \bar{Q}[A] + \varepsilon = M_0 + \varepsilon$. By Lemma 10.1, this implies $M_T \in \mathcal{C}(M_0 + \varepsilon)$ and so by definition $M_T \leq V_T$ for some $V_T \in \mathcal{V}(M_0 + \varepsilon)$. Therefore $E[hM_T] \leq z(M_0 + \varepsilon)$ by definition of $\mathcal{D}(z)$. However, $E[zZ_T^{\bar{Q},P} M_T] = \bar{E}_\bar{Q}[zM_T] = zM_0$. It follows $E[(h - zZ_{\bar{Q},P} M_T) \leq \varepsilon]$. But clearly
\[ E[(h - zZ_{\bar{Q},P}^{\bar{Q},P}) M_T] \geq E[(h - zU_T) 1_A] \geq E[(h - zU_T) 1_{A_\delta}] \geq \delta P[A_\delta]. \]

Hence $\delta P[A_\delta] \leq \varepsilon \varepsilon$ but sending $\varepsilon \to 0$ implies $P[A_\delta] = 0$. This is a contradiction.