Mathematical Finance

Solution sheet 9

Solution 9.1

(a) Because U is concave and C^1, we have that $U'(x)$ is decreasing. Fix $x_0 > 0$. The mapping $x \mapsto \frac{U(x)-U(x_0)}{x-x_0}$ is also decreasing. By the mean value theorem, we have

$$\frac{U(x) - U(x_0)}{x - x_0} = U'(x')$$

for some $x' \in [x_0, x]$. Hence, for all $x > x_0$,

$$\frac{x - x_0}{U(x) - U(x_0)} U'(x) = \frac{U'(x)}{U'(x')} \leq 1.$$

If $U(\infty) \leq 0$, then clearly $AE_{+\infty}(U) \leq 0$. So assume $U(\infty) > 0$ and $U(x_0) > 0$. Then for $x > x_0$, we have

$$\frac{x}{U(x)} \leq \frac{x}{U(x) - U(x_0)}.$$

Together with $\frac{x-x_0}{x} \to 1$ as $x \to \infty$, we obtain

$$\limsup_{x\to\infty} \frac{xU'(x)}{U(x)} \leq \limsup_{x\to\infty} \frac{x - x_0}{U(x) - U(x_0)} U'(x) \leq 1.$$

(b) By definition, we have

$$AE_{+\infty}(U) = \inf\{\gamma > 0 : \exists x_0 \text{ s.t. } U'(x) < \gamma U(x)/x,\ \forall x \geq x_0\} =: R. \tag{1}$$

So we need to show

$$R = \inf\{\gamma > 0 : \exists x_0 \text{ s.t. } U(\lambda x) < \lambda^\gamma U(x)\ \forall \lambda > 1, x \geq x_0\}.$$

Set $F(\lambda) := U(\lambda x)$ and $G(\lambda) = \lambda^\gamma U(x)$ for $\lambda > 1$.

"$\geq$": Fix $x > x_0$, $\gamma > 0$ satisfying the property of R. Clearly $F(1) = U(x) = G(1)$ and $F'(1) = xU'(x) < \gamma U(x) = G'(1)$. Hence $F(\lambda) < G(\lambda)$ on $(1, 1+\varepsilon)$ for some $\varepsilon > 0$. Set $\tilde\lambda := \inf\{\lambda > 1 : F(\lambda) = G(\lambda)\}$. We only need to argue that $\tilde\lambda = \infty$. But if not, we must have $F'(\tilde\lambda) \geq G'(\tilde\lambda)$ which contradicts to

$$F'(\tilde\lambda) = xU'(\tilde\lambda x) \underbrace{<}_{(1)} \frac{\gamma}{\tilde\lambda} U(\tilde\lambda x) = \frac{\gamma}{\tilde\lambda} F(\tilde\lambda) = \frac{\gamma}{\tilde\lambda} G(\tilde\lambda) = G'(\tilde\lambda).$$

"$\leq$": Let γ be such that it satisfies the property on RHS in (1). Then $F(1) = G(1)$ and $F(\lambda) < G(\lambda)$ for all $\lambda > 1$ give $F'(1) \leq G'(1)$. Hence

$$U'(x) = \frac{F'(1)}{x} \leq \frac{G'(1)}{x} = \gamma \frac{U(x)}{x}.$$

(c) We first argue that $AE_+(u) \leq AE_+(U)$. WLOG, we may assume that $U(\infty) > 0$. Let $\gamma > AE_+(U)$. Then by part (b), there is $x_0 > 0$ such that

$$U(\lambda x) < \lambda^\gamma U(x) \text{ for } \lambda > 1, x > x_0. \tag{2}$$

So it is enough to show that there is $x_1 > 0$ such that

$$u(\lambda x) < \lambda^\gamma u(x) \text{ for } \lambda > 1, x > x_1. \tag{3}$$

Assume for the moment that $x_0 = 0$. Then using $\frac{1}{\lambda} I(h^*_{\lambda x}) \in \mathcal{C}(x)$ gives

$$u(\lambda x) = E[U(I(h^*_{\lambda x}))] \leq E\left[\lambda^\gamma U\left(\frac{I(h^*_{\lambda x})}{\lambda}\right)\right] \leq \lambda^\gamma u(x) \text{ for all } x > 0.$$

For the general case, we define

$$\tilde{U} := \begin{cases} c_1 \frac{x^\gamma}{\gamma} & x \leq x_0 \\ c_2 + U(x) & x \geq x_0 \end{cases}$$

which is C^1 on $(0, \infty)$. Then $\tilde{U}$ satisfies (2) with $x_0 = 0$. Hence the corresponding $\tilde{u}$ satisfies (3) with $x_1 = 0$. Thus taking the infimum over γ yields $AE_+(\tilde{u}) \leq AE_{+\infty}(U)$. Now we show that $\tilde{u}$ and and u are close to each other for large x. Then by the hint, the proof is complete. Clearly there exists $K > 0$ such that

$$U(x) - K \leq \tilde{U}(x) \leq U(x + x_0) + K, \text{ for all } x > 0,$$

hence we also have

$$u(x) - K \leq \tilde{u}(x) \leq u(x + x_0) + K.$$

Thus there exists $C > 0$ and $x_2 > 0$ such that

$$u(x) - C \leq \tilde{u}(x) \leq u(x) + C \text{ for } x \geq x_2.$$

This proves $AE_{+\infty}(u) = AE_{+\infty}(\tilde{u}) \leq AE_{+\infty}(U) < 1$. To show $u'(\infty) = 0$, we argue by assuming that $u'(\infty) = c > 0$ and seeking a contradiction. Then since u is strictly concave, we have that u' is decreasing and for large x

$$\frac{xu'(x)}{u(x)} \geq \frac{xu'(\infty)}{u(x)} \underbrace{\sim}_{\text{l'Hospital's rule}} \frac{u'(\infty)}{u'(x)} \to \frac{c}{c} = 1.$$

This is a contradiction.

Solution 9.2 By definition, we have $J_n(h) \geq U(f) - fh$ for all $f \in \mathcal{B}_n$. So $E[J_n(h)] \geq E[U(f) - fh]$ and taking the supremum yields "$\geq$". Now for fixed h, the mapping $x \mapsto U(x) - xh$ is concave and so the maximiser x^* is either in $(0, n)$ or $> n$. In the first case, we have $x = I(h)$ and in particular $\max_{x \in [0,n]}(U(x) - xh) = U(I(h)) - hI(h)$. In the second case, we must have $\max_{x \in [0,n]}(U(x) - xh) = U(n) - nh$. Then obviously, $f^* := I(h) \wedge n \in \mathcal{B}_n$ and maximizes $x \mapsto U(x) - xh$ on $[0, n]$ P-a.s. Thus the claim is established.

Solution 9.3

(a) First observe that by assumption K is bounded, thus $E[\exp(\frac{1}{2}\langle -\lambda \bullet M \rangle_T)] = E[\exp(\frac{1}{2}K_T)] < \infty$. By Novikov's condition, $\hat{Z}$ is therefore a martingale > 0 on $[0, T]$ and $\hat{P}$ is an equivalent probability measure. To show that S is a local martingale under $\hat{P}$, we compute

$$
\begin{aligned}
\mathrm{d}(\hat{Z}S) &= \hat{Z}\,\mathrm{d}S + S\,\mathrm{d}\hat{Z} + \mathrm{d}\langle \hat{Z}, S \rangle \\
&= \hat{Z}\,\mathrm{d}M + \hat{Z}\lambda\,\mathrm{d}\langle M \rangle - S\hat{Z}\lambda\,\mathrm{d}M - \hat{Z}\lambda\,\mathrm{d}\langle M \rangle \\
&= (\hat{Z} - S\hat{Z}\lambda)\,\mathrm{d}M.
\end{aligned}
$$

(b) We compute, for any $p \in \mathbb{R}$,

$$
\begin{aligned}
\hat{Z}_T^p &= \exp\left(-p\lambda \bullet M_T - \frac{1}{2}p\lambda^2 \bullet \langle M \rangle_T \right) \\
&= \exp\left(-p\lambda \bullet M_T - \frac{1}{2}p^2\lambda^2 \bullet \langle M \rangle_T \right) \exp\left(\frac{1}{2}(p^2 - p)\lambda^2 \bullet \langle M \rangle_T \right) \\
&= \mathcal{E}(-p\lambda \bullet M)_T \exp((p^2 - p)K_T).
\end{aligned}
$$

So $E[\hat{Z}_T^p] \leq CE[\mathcal{E}(-p\lambda \bullet M)_T] < \infty$ by the fact that $\mathcal{E}(-p\lambda \bullet M)$ is a supermartingale and the boundedness of K again.