Mathematical Finance

Solution sheet E

Solution E.1

(a) First, we assume that A is increasing. By localization, we may assume that A is integrable. So $E[A_\infty] < \infty$ and A is a submartingale of class D. So by the Doob–Meyer decomposition, there exists a unique predictable RCLL process A^p null at 0 such that $A - A^p$ is a UI martingale.

If A is of FV with integrable variation, we can write $A = B - C$ for some increasing integrable processes B, C. Set $A^p := B^p - C^p$ and we get that $A - A^p = B - B^p + C - C^p$ is a UI martingale.

If $\tilde{A}^p$ is another process satisfying all the required properties, then $A^p - \tilde{A}^p = (A^p - A) - (\tilde{A}^p - A)$ is a predictable local martingale of FV null at 0. So this must be continuous (all predictable local martingales are continuous) and hence constant (all continuous local martingales of finite variation are constant). Therefore, we have the uniqueness.

(b) Let M be a local martingale. Denote by V the variation process of M. By localization, we may assume that $M \in \mathcal{H}^1_0$. Set $\tau_n = \inf\{t \geq 0 : V_t > n\}$; then clearly (τ_n) are stopping times with $\tau_n \uparrow \infty$. Using $\triangle V = |\triangle M|$, we have for each $t > 0$ that $|\triangle V_t| \leq |M_{t-}| + |M_t|$; in particular,
$|V_{\tau_n}| \leq n + |M_{\tau_n -}| + |M_{\tau_n}| \leq 2n + |M_{\tau_n}| \in L^1$ because $|M_{\tau_n -}| \leq V_{\tau_n -} \leq n$ and $M \in \mathcal{H}^1_0$.

Solution E.2

(a) The uniqueness again follows from the fact that any predictable local martingale of FV must be constant like in Exercise E.1 (a). Clearly $\langle M, N\rangle$ is a predictable RCLL process of FV. Also, $MN - [M, N] = M_- \bullet N + N_- \bullet M$ is a local martingale as a sum of two stochastic integrals of locally bounded processes w.r.t. local martingales. By definition, $[M, N] - [M, N]^p$ is a local martingale. So $MN - \langle M, N\rangle = (MN - [M, N]) + ([M, N] - [M, N]^p)$ is a local martingale.

(b) By the polarization identity, we only need to construct $\langle M\rangle := \langle M, M\rangle$. By localization, we assume that $M \in \mathcal{H}^2$. Doob's inequality shows that $E[\sup_{s\geq 0} M_s^2] \leq 4E[M_\infty^2]$ and so M^2 is a submartingale of class D. Applying the Doob–Meyer decomposition to M^2 yields a predictable increasing process $\langle M\rangle$ null at 0 such that $M^2 - \langle M\rangle$ is a martingale. Finally, $[M]^p = \langle M\rangle$ is a consequence of part (a). Of course, then, $\langle M\rangle$ is a predictable RCLL process null at 0 such that $M^2 - \langle M \rangle$ is a martingale. By uniqueness in (a), this implies that $[M]^p$ equals $\langle M \rangle$ constructed in (b).

(c) "$\Longrightarrow$" By localization, we may assume $M \in \mathcal{H}_0^2$. Note that $[M] - \langle M \rangle$ is a local martingale null at 0. Fix $t > 0$. Choose a localizing sequence (τ_n) such that $([M] - \langle M \rangle)^{\tau_n}$ is a martingale. In particular, we have

$$E\Big[[M]_{t\wedge\tau_n} - \langle M \rangle_{t\wedge\tau_n}\Big] = 0$$

for each $n \in \mathbb{N}$. But we know from the proof of part (b) that $M^2 - \langle M \rangle$ is a martingale and hence $E[\langle M \rangle_{t\wedge\tau_n}] < \infty$ and $E[[M]_{t\wedge\tau_n}] = E[\langle M \rangle_{t\wedge\tau_n}]$ for each $n \in \mathbb{N}$. Sending $n \to \infty$ and using monotone convergence on both sides yields $E[[M]_t] = E[\langle M \rangle_t]$. Since $M \in \mathcal{H}_0^2$, $E[\langle M \rangle_\infty] = E[M_\infty^2] < \infty$. So we have $E[[M]_\infty] < \infty$ which shows that $[M]$ has integrable variation.

"$\Longleftarrow$" Let (τ_n) be a localizing sequence such that $(M^2 - [M])^{\tau_n}$ is a martingale and $[M]^{\tau_n}$ has integrable variation. Then

$$\sup_{t\geq 0} E[(M_t^{\tau_n})^2] = \sup_{t\geq 0} E[M_{t\wedge\tau_n}^2] = \sup_{t\geq 0} E[[M]_{t\wedge\tau_n}] = E[[M]_{\tau_n}] < \infty.$$

(d) This is a direct consequence of Exercise 4.4 (d).

(e) Clearly M is an adapted RCLL process of FV. Also notice that $\triangle M = \triangle N$ and $\triangle N$ can be only 0 or 1. So by part (d) and we have

$$[M]_t = \sum_{s\leq t} (\triangle M_s)^2 = \sum_{s\leq t} (\triangle N_s)^2 = N_t.$$

By part (a), we know that $\langle M \rangle = [M]^p = N^p$. For $s \leq t$,

$$E[M_t - M_s | \mathcal{F}_s] = E[N_t - N_s | \mathcal{F}_s] - \lambda(t-s) = E[N_t - N_s] - \lambda(t-s) = 0.$$

Together with obvious adaptedness and integrablility of M, we see that M is a martingale. Clearly $(\lambda t)_{t\geq 0}$ is predictable RCLL and of FV null at 0. So by the uniqueness we have $N_t^p = \lambda t$ and hence $\langle M \rangle_t = \lambda t$.

Solution E.3 Write $X_t = \sum_{k=1}^{N_t} Y_k$, where N is a Poisson process with rate λ and $(Y_k)_{k\in\mathbb{N}}$ a sequence of random variables independent of N such that the Y_i are i.i.d. with distribution ν. Note that W, N and $(Y_k)_{k\in\mathbb{N}}$ are independent. If R is a martingale, then in particular $E[R_T] = 0$, and hence $E[X_T] = -aT$. Since $E[X_T] = \lambda T E[Y_1]$, this gives $E[Y_1] = -\frac{a}{\lambda}$. Since $\mathcal{E}(R)$ is a nonnegative local martingale and hence a supermartingale, it suffices to show that $E[\mathcal{E}(R)_T] = 1$. Using the formula for $\mathcal{E}(R)$, the fact that X is a simple jump process, i.e. $X_t = \sum_{0<s\leq t} \Delta X_s$

and hence $\exp(X_t)\prod_{0<s\leq t}\exp(-\Delta X_s) = 1$, the fact that W, N and $(Y_k)_{k\in\mathbb{N}}$ are independent and that $\Delta Y_s = \Delta X_s$ gives

$$\begin{aligned} E[\mathcal{E}(R)_T] &= E\left[\exp(aT)\exp\left(\sigma W_T - \frac{1}{2}\sigma^2 T\right)\prod_{k=1}^{N_T}(1+Y_k)\right] \\ &= \exp(aT)E\left[\left(1-\frac{a}{\lambda}\right)^{N_T}\right] = \exp(aT)\exp\left(\left(-\frac{a}{\lambda}\right)\lambda T\right) \\ &= 1. \end{aligned}$$

Remark: One can show in general that if R is a Lévy process and a local martingale, then $\mathcal{E}(R)$ is a martingale.