Assignment 19

Normality and separability

1. Let $f \in k[X]$ be a monic polynomial which splits into linear factors over k. Suppose that $\sigma \in \mathrm{Aut}(k)$ fixes each root of f. Prove that σ fixes all the coefficients of f.

2. Let E/k be a splitting field of $f \in k[X]$ and consider an extension k' of k and the splitting field E' of f over k'. Show that each $\sigma \in \mathrm{Aut}(E'/k')$ satisfies $\sigma(E) = E$ and that the resulting homomorphism

$$\begin{aligned} \varphi\colon \mathrm{Aut}(E'/k') &\longrightarrow \mathrm{Aut}(E/k) \\ \sigma &\longmapsto \sigma|_E \end{aligned}$$

   is injective.

3. Let E/k be a finite field extension and let $\bar{E}$ be an algebraic closure of E (and thus of k).

   (a) Prove that the following are equivalent:

      (i) For every k-homomorphism $\varphi\colon E \to \bar{E}$ we have $\varphi(E) \subset E$.

      (ii) Every irreducible polynomial $f \in k[X]$ with a root in E splits into linear factors over E.

      (iii) E is the splitting field of some polynomial $f \in k[X]$.

      Hint. Prove (ii)⇒(iii)⇒(i)⇒(ii) and use the fact that every k-homomorphism $K \to \bar{K}$ can be extended to a k-homomorphism $K(a) \to \bar{K}$ for any $a \in \bar{K}$.

   (b) Suppose that the minimal polynomial over k of any element in E has distinct roots in $\bar{E}$. Prove that E/k is Galois if and only if every irreducible polynomial $f \in k[X]$ with a root in E splits into linear factors over E.

4. Show that $\mathrm{Aut}(\mathbb{R}) = \{\mathrm{id}_{\mathbb{R}}\}$.