D-MATH Algebra II FS19
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# Assignment 23

## Solvability by Radicals. Recap.

1. Prove that the groups $S_2, S_3$ and $S_4$ are solvable.

2. Let $k$ be a field and $n = 2d$ a positive even integer. Let $f = \sum_{j=0}^{n} a_j X^j \in k[X]$ be a monic polynomial of degree $n$ without multiple roots and suppose that $f$ has no root in $k$. Suppose moreover that $f$ is palindromic, that is, $a_j = a_{n-j}$ for each $j \in \{0, \dots, d\}$. Let $E$ be the splitting field of $f$.

   (a) Prove that $x \mapsto \frac{1}{x}$ is a well-defined bijection on the set of roots of $f$.

   (b) Deduce that $\# \operatorname{Gal}(E/k)$ divides $2^d d!$.

3. For each of the following polynomials, determine the Galois group of its splitting field:

   (a) $X^4 + 2X^3 + X^2 + 2X + 1 \in \mathbb{Q}[X]$ *Hint.* Exercise 2

   (b) $X^4 + X + 1 \in \mathbb{F}_2[X]$

   (c) $X^5 + \frac{5}{4}X^4 - \frac{5}{21} \in \mathbb{Q}[X]$
   *Hint.* Show that the polynomial has precisely three real roots and deduce that the Galois group contains a transposition and a 5-cycle.

   (d) $X^{81} - t \in \mathbb{F}_3(t)[X]$