Assignment 25

Finite fields

1. Let k be a field.
   (a) Show that k is an extension of a field k_0, called *prime field*, given by $k_0 = \mathbb{Q}$ if char $(k) = 0$ $k_0 = \mathbb{F}_p$ and if char $(k) = p > 0$.
   (b) Show that any field homomorphism restricts to the identity on the prime field.
2. We say that a field k is *perfect* if every algebraic field extension of k is separable.
   (a) Prove that k is perfect if and only if every irreducible polynomial in $k[X]$ is *separable*, i.e. has no multiple roots.
   (b) Let $f \in k[X]$ be an irreducible polynomial. Show that f is separable if and only if its derivative is nonzero.
   (c) For f as in Part (b), show that the derivative of f is zero if and only if char $(k) = p > 0$ and $f(X) = g(X^p)$ for some irreducible $g \in k[X]$.
   (d) Suppose that char $(k) = p > 0$. Prove that k is perfect if and only if the Frobenius homomorphism $\varphi \colon k \to k$, $x \mapsto x^p$ is surjective.
   (e) Deduce that fields of characteristic zero and finite fields are perfect.
3. Let k be a finite field and consider a finite field extension $k(\alpha, \beta)/k$ such that $k(\alpha) \cap k(\beta) = k$ (inside an algebraic closure of k). Prove that $k(\alpha, \beta) = k(\alpha + \beta)$.

   Hint. Study the cardinality of the involved fields.
4. Give a detailed proof of Wedderburn's theorem: *Every finite skew-field is a field.*