Assignment 26

Representations of finite groups

1. Show that the image of a one-dimensional representation of a finite group is a cyclic group.

2. Let H be a subgroup of index 2 of a group G, and let $\sigma\colon H \to \mathrm{GL}(V)$ be a representation. Let a be an element in $G \setminus H$. Define a *conjugate* representation ${}^a\sigma\colon H \to \mathrm{GL}(V)$ by the rule ${}^a\sigma(h) = \sigma(a^{-1}ha)$. Prove that

   (a) The conjugate representation ${}^a\sigma$ is indeed a representation of H.

   (b) If σ is the restriction to H of a representation of G, then ${}^a\sigma$ is isomorphic to σ.

   (c) If b is another element of $G \setminus H$, then the conjugate representation ${}^b\sigma$ is isomorphic to ${}^a\sigma$.

3. Let $\rho\colon G \to \mathrm{GL}(V)$ be a representation of a finite group on a real vector space V. Prove the following:

   (a) There exists a G-invariant, positive definite, symmetric form $\langle , \rangle$ on V.

   (b) The representation ρ is a direct sum of irreducible representations.

4. Consider the representation ρ of $\mathbb{Z}$ on $\mathbb{C}^2$ defined by $\rho(1) = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$.

   (a) Find a proper invariant subspace.

   (b) Show that ρ is not a direct sum of irreducible representations.

A *character table* consists of rows corresponding to irreducible group representations, and columns corresponding to conjugacy classes of group elements. The entries are values of the character of the representation in a given row on the conjugacy class in a given column.

5. Determine the character table for the Klein four group.

6. Consider the dihedral group D_5, and its cyclic subgroup C_5.

   (a) Determine the character table of D_5 and of C_5.

   (b) Decompose the restriction of each irreducible character of D_5 into irreducible characters of C_5.

7. The quaternion group Q is the group $Q = \langle i, j, k \mid i^2 = j^2 = k^2 = -1, ijk = -1 \rangle$

   (a) Find a subgroup of $\mathrm{GL}_2(\mathbb{C})$ isomorphic to Q and determine the order of Q.

   (b) Determine the conjugacy classes of Q.

   (c) Show that any subgroup of Q is normal.

   (d) Determine the character table of Q.

No hand-in. *Enjoy your semester break!*