Solution 21

Constructions with straightedge and compass.

1. (a) Express $\cos\frac{\pi}{12}$ in terms of real square roots.
   (b) Prove that an angle α can be trisected if and only if $4X^3 - 3X - \cos\alpha$ is reducible over $\mathbb{Q}(\cos\alpha)$.

   Solution:

   (a) Set $\beta := \frac{\pi}{12}$. We have $\cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$ and thus

$$\frac{\sqrt{3}}{2} = \cos(2\beta) = 2\cos^2(\beta) - 1,$$

   so

$$\cos(\beta) = \sqrt{\frac{\sqrt{3}+2}{4}} = \frac{\sqrt{2+\sqrt{3}}}{2}.$$

   (b) This follows immediately from the identity $\cos(\alpha) = 4\cos^3(\alpha/3) - 3\cos(\alpha/3)$.

2. (Constructible polygons)

   (a) Is the regular 9-gon constructible?
   (b) Deduce that the regular 15-gon is constructible from the fact that the regular 3-gon and the regular 5-gon are.
   (c) More generally, prove that if m and n are coprime and both the m-gon and the n-gon are constructible, then so is the mn-gon.

   Solution: We repeatedly use the fact that a regular n-gon is constructible if and only if its associated angle $2\pi/n$ is constructible.

   (a) We know from the lecture that $\cos(\pi/9)$ is not constructible. Using the trigonometric identity $\cos^2(\alpha) = \frac{1+\cos(2\alpha)}{2}$, we write

$$\cos(2\pi/9) = 2\left(\cos(\pi/9)\right)^2 - 1.$$

   Since $\cos(\pi/9)$ is not constructible, neither is its square and it follows that $\cos(2\pi/9)$ is not constructible.

(b) If the regular 3- and 5-gons are constructible, then their angles $2\pi/3$ and $2\pi/5$ are. Thus so is the angle $2 \cdot 2\pi/5 - 2\pi/3 = 2\pi/15$ of the regular 15-gon.

(c) With coprime m and n, set $\alpha := 2\pi/mn$, and assume we can construct $\beta = 2\pi/n = m\alpha$ and $\gamma = 2\pi/m = n\alpha$. Since m and n are coprime, by Euclidean division there exist integers u and v such that $1 = um + vn$. Thus, $\alpha = u\beta + v\gamma$ can also be constructed.

3. Is it possible to construct a square whose area is that of a given triangle?

Solution: Let Δ denote the given triangle. As $\text{area}(\Delta) = \frac{1}{2}bh$, where b refers to the base of Δ and h to its height, we construct a length s such that $s^2 = \frac{1}{2}bh$. We know how to mark off lengths, here $b/2$ and h, onto a constructed line so that the two segments are adjacent. Then take the circle passing through the endpoints of diameter $b/2 + h$, and then the triangle inscribed in this circle with vertices the endpoints.

We can now check that the height s of this triangle is exactly the length we are looking for by checking

$$\frac{s}{h} = \frac{b/2}{s}.$$

Denote the angles α, β, γ, where $\beta = \pi/2$ breaks down in $\beta = \beta_1 + \beta_2$ for the triangle on the left and the triangle on the right. We then observe

$$\frac{s}{h} = \tan\gamma = \tan(\pi/2 - \beta_2) = \tan(\beta_1) = \frac{b/2}{s}.$$

4. Show that a root α of $f(X) := X^4 - 4X + 2 \in \mathbb{Q}[X]$ generates a degree 4 extension of $\mathbb{Q}$ but is not constructible.

Hint: The splitting field of f over $\mathbb{Q}$ has Galois group S_4.

Solution: The polynomial f is irreducible by the Eisenstein criterion at $p = 2$, thus any root of f generates a degree 4 extension of $\mathbb{Q}$.

Let E be the splitting field of f over $\mathbb{Q}$.

Version 1. If the roots of f were constructible, then all the elements of E would be constructible. Let H be a 2-Sylow subgroup of $\text{Gal}(E/\mathbb{Q})$. Then the fixed field E^H has odd degree over $\mathbb{Q}$, and so the elements of $E^H/\mathbb{Q}$ cannot be constructible.

Version 2. By Corollary 7 of the lecture, a number $\alpha \in \mathbb{C}$ is constructible if and only if it lies in a tower of degree 2 extensions. Suppose there exists a tower of quadratic extensions $\mathbb{Q}[\alpha] \supset F \supset \mathbb{Q}$. By Galois theory, the groups $\text{Gal}(E/F) \supset \text{Gal}(E/\mathbb{Q}[\alpha])$ have orders 12 and 6 respectively. As $\text{Gal}(E/\mathbb{Q}) = S_4$, we must have $\text{Gal}(E/F) = A_4$. But A_4 has no subgroup of order 6 – contradiction.