Solution 26

Representations of finite groups

1. Show that the image of a one-dimensional representation of a finite group is a cyclic group.

   Solution: Let G be a finite group. A representation of dimension 1 of G is an homomorphism $\rho : G \to GL_1(\mathbb{C})$. The group $GL_1(\mathbb{C})$ coincides with the multiplicative group $\mathbb{C}^\times$. Let z be an element in the image of ρ. Since G is finite, we get that $z^n = 1$ for some n, in particular the image of ρ is a subgroup of the group S^1 of complex numbers of modulus one. We saw in class that all finite subgroups of S^1 are cyclic and this finishes the proof.

2. Let H be a subgroup of index 2 of a group G, and let $\sigma\colon H \to \mathrm{GL}(V)$ be a representation. Let a be an element in $G \setminus H$. Define a *conjugate* representation ${}^a\sigma\colon H \to \mathrm{GL}(V)$ by the rule ${}^a\sigma(h) = \sigma(a^{-1}ha)$. Prove that

   (a) The conjugate representation ${}^a\sigma$ is indeed a representation of H.

   (b) If σ is the restriction to H of a representation of G, then ${}^a\sigma$ is isomorphic to σ.

   (c) If b is another element of $G \setminus H$, then the conjugate representation ${}^b\sigma$ is isomorphic to ${}^a\sigma$.

   Solution:

   (a) In order to check that ${}^a\sigma$ is well defined, let us notice that, since H has index 2 in G, H is normal, hence for every element $h \in H$ the element $a^{-1}ha$ still belongs to H, in particular the value $\sigma(a^{-1}ha)$ is well defined. Let us now check that ${}^a\sigma$ is a representation. In order to do this, it is enough to verify that ${}^a\sigma$ is an homomorphism in GL_n for some n. In particular, since σ is a representation in $GL(V)$, the image of ${}^a\sigma$ is also contained in $GL(V)$, hence we only have to verify that ${}^a\sigma(gh) = {}^a\sigma(g){}^a\sigma(h)$. But this follows from the definition:

   $${}^a\sigma(gh) = \sigma(a^{-1}gha) = \sigma((a^{-1}ga)(a^{-1}ha)) = \sigma(a^{-1}ga)\sigma(a^{-1}ha) = {}^a\sigma(g){}^a\sigma(h).$$

(b) Assume that σ is the restriction of a representation of G, and let $A \in GL(V)$ be the element $\sigma(a)$. In order to show that σ and ${}^a\sigma$ are conjugate we need a linear isomorphism $L : V \to V$ such that, for any $h \in H$, $\sigma(h)(Lv) = L({}^a\sigma(h)v)$. The linear map A is such an isomorphism, indeed for any $v \in V$

$$\sigma(h)Av = AA^{-1}\sigma(h)Av = A\sigma(a^{-1})\sigma(h)\sigma(a)v = A\sigma(a^{-1}ha) = A\sigma(h)v.$$

(c) Since b is another element of G that doesn't belong to H, and since H has index 2 in G, then there exists an element h in H such that $b = ah$. Now we have, for every $g \in H$, that ${}^b\sigma(g) = \sigma(h^{-1}a^{-1}gah) = \sigma(h^{-1})^a\sigma(g)\sigma(h)$. In particular this implies that, for every $g \in H$, we have $\sigma(h)^b\sigma(g) = {}^a\sigma(g)\sigma(h)$ and the linear map $\sigma(h) : V \to V$ gives an isomorphism of the representations ${}^a\sigma$ and ${}^b\sigma$.

3. Let $\rho\colon G \to \mathrm{GL}(V)$ be a representation of a finite group on a real vector space V. Prove the following:

   (a) There exists a G-invariant, positive definite, symmetric form $\langle , \rangle$ on V.

   (b) The representation ρ is a direct sum of irreducible representations.

   Solution:

   (a) Let us fix a positive definite, symmetric bilinear form $[\ ,\ ]$ on V. (To find such a form it is enough to fix an isomorphism of V with $\mathbb{R}^n$ and consider the standard positive definite symmetric bilinear form on $\mathbb{R}^n$). And let us define the averaged form by setting

$$\langle v, w\rangle = \frac{1}{|G|}\sum_{g\in G}[\rho(g)v, \rho(g)w].$$

The form is symmetric, positive definite and G-invariant. The fact that it is symmetric follows from the symmetry of $[\ ,\ ]$:

$$\langle v, w\rangle = \frac{1}{|G|}\sum_{g\in G}[\rho(g)v, \rho(g)w] = \frac{1}{|G|}\sum_{g\in G}[\rho(g)w, \rho(g)v] = \langle v, w\rangle.$$

To check that the form is positive, it is enough to check that $\langle v, v\rangle > 0$, but we have

$$\langle v, v\rangle = \frac{1}{|G|}\sum_{g\in G}[\rho(g)v, \rho(g)v] > 0$$

since the latter expression is a sum of positive numbers, the verification of the G invariance follows rearranging the summation, once one notices that for any element $h \in G$, right multiplication by h gives a permutation of G:

$$\begin{aligned}\langle \rho(h)v, \rho(h)w\rangle &= \tfrac{1}{|G|}\textstyle\sum_{g\in G}[\rho(g)\rho(h)v, \rho(g)\rho(h)w]\\ &= \tfrac{1}{|G|}\textstyle\sum_{gh\in G}[\rho(gh)v, \rho(gh)w] = \langle v, w\rangle.\end{aligned}$$

(b) If ρ is irreducible, there is nothing to prove. Assume that ρ is not irreducible and let $W < V$ a ρ-invariant subspace. We claim that the orthogonal of W with respect to the form $\langle\ ,\ \rangle$ defined in the previous part is also ρ-invariant. Indeed it is enough to check that if $z \in W^\perp$ and $g \in G$, then $\rho(g)z$ is in $W^\perp$, or equivalently for every $w \in W$ $\langle \rho(g)z, w\rangle = 0$. But this is true since

$$\langle \rho(g)z, w\rangle = \langle \rho(g^{-1})\rho(g)z, \rho(g^{-1})w\rangle = \langle z, \rho(g^{-1})w\rangle = 0$$

Here the first equality is due to the fact that $\langle\ ,\ \rangle$ is ρ-invariant, the second one to the fact that ρ is a representation, the third to the fact that $\rho(g^{-1})w \in W$ since W is G-invariant, and z belongs to $W^\perp$. This implies that ρ splits as a direct sum of two representations ρ', ρ''. Since V is finite dimensional the conclusion follows by induction.

4. Consider the representation ρ of $\mathbb{Z}$ on $\mathbb{C}^2$ defined by $\rho(1) = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$.

(a) Find a proper invariant subspace.

(b) Show that ρ is not a direct sum of irreducible representations.

Solution:

(a) Clearly the subspace of $\mathbb{C}^2$ generated by the first element of the standard basis is invariant under the representation ρ: indeed any element in $\mathbb{C}e_1$ is of the form $\left(\begin{smallmatrix} a \\ 0 \end{smallmatrix}\right)$ for some $a \in \mathbb{C}$. Moreover, for any $n \in \mathbb{Z}$, we have that $\rho(n)$ is the matrix $\left(\begin{smallmatrix} 1 & n \\ & 1 \end{smallmatrix}\right)$ and so $\left(\begin{smallmatrix} 1 & n \\ & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} a \\ 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} a \\ 0 \end{smallmatrix}\right)$.

(b) Assume by contradiction that ρ is the direct sum of irreducible representations. Then there would exist another vector $w \in \mathbb{C}^2$ that doesn't belong to $\mathbb{C}e_1$, and such that $\rho(n)w = w$ for every $n \in \mathbb{Z}$. But since $\rho(1)$ has no other eigenvector apart from e_1, this is clearly not possible.

5. Determine the character table for the Klein four group.

Solution: The Klein four group K is isomorphic to the product $\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$. Since it is abelian, its conjugacy classes coincide with the four group elements. Let a and b be generators of K. In order to compute the character table for K it is enough to determine four distinct non-isomorphic one-dimensional representations of K. If ρ is a one-dimensional representation of K, the image of a generator of K must be an element of order two in $\mathbb{C}^\times$, i.e. an element of the set $\{\pm 1\}$. Moreover, for any such choice we get a non-isomorphic representation of K. This implies that the character table of K is

K	1	a	b	ab
ρ_1	1	1	1	1
ρ_a	1	-1	1	-1
ρ_b	1	1	-1	-1
ρ_{ab}	1	-1	-1	1

6. Consider the dihedral group D_5, and its cyclic subgroup C_5.

   (a) Determine the character table of D_5 and of C_5.

   (b) Decompose the restriction of each irreducible character of D_5 into irreducible characters of C_5.

   Solution:

   (a) Let ω denote a primitive fifth root of unity in $\mathbb{C}^*$. There are five conjugacy classes in C_5, corresponding to the five elements. In order to determine a representation of C_5, it is enough to describe the image of the generator that is going to be a fifth root of unity, hence a power of ω. In particular we get that the character table of C_5 is

C_5	0	1	2	3	4
τ_1	1	1	1	1	1
τ_ω	1	ω	ω^2	ω^3	ω^4
τ_{ω^2}	1	ω^2	ω^4	ω	ω^3
τ_{ω^3}	1	ω^3	ω	ω^4	ω^2
τ_{ω^4}	1	ω^4	ω^3	ω^2	ω

   In order to compute the conjugacy classes in D_5 let us notice that, since 5 is odd, all reflections are conjugate, hence form a conjugacy class C_y, moreover the rotations come in three different conjugacy classes: $\{1\}, \{x, x^4\}, \{x^2, x^3\}$. This implies that we have to exhibit five different irreducible representations of D_5. We will denote by ρ_1 the trivial representation. Let us consider the subgroup C_5 of D_5. It is a normal subgroup and the quotient $D_5/C_5 = \mathbb{Z}/2\mathbb{Z}$. This gives another one dimensional (hence irreducible) representation of D_5, the sign representation. We will denote it by sign.

   Let us now consider the standard representation of D_5 as a subgroup of $O_2(\mathbb{R})$. We showed in class that the group D_5 is isomorphic to the subgroup of O_2 generated by a reflection and a rotation of angle $2\pi/5$. The matrix expression for a rotation of angle $2\pi/5$ is

$$R_x = \begin{pmatrix} \cos(2\pi/5) & \sin(2\pi/5) \\ -\sin(2\pi/5) & \cos(2\pi/5) \end{pmatrix}$$

and the matrix expression for the reflection along the x axis is $R_y = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$. Interpreting these two matrices as complex matrices, and letting them act on $\mathbb{C}^2$ we get a complex representation of D_5 that we will denote by ρ_ω. It is well known that if ω can be chosen to be $\cos(2\pi/5) + i\sin(2\pi/5)$. In particular it is easy to compute the character of ρ_ω (see the table below). Since the character has norm one, we get that the representation is irreducible. The last irreducible representation ρ_{ω^2} of D_5 is obtained in a similar manner, by defining $\rho_{\omega^2}(y) = R_y$ and $\rho_{\omega^2}(x) = R_x^2$. Computing the character of this representation one can easily check that ρ_{ω^2} is irreducible and ρ_{ω^2} is not isomorphic to ρ_ω. This leads to the character table for D_5.

D_5	$\{1\}$	$\{x, x^4\}$	$\{x^2, x^3\}$	C_y
ρ_1	1	1	1	1
sign	1	1	1	-1
ρ_ω	2	$\omega + \bar{\omega}$	$\omega^2 + \bar{\omega}^2$	0
ρ_{ω^2}	2	$\omega^2 + \bar{\omega}^2$	$\omega + \bar{\omega}$	0

(b) The restriction to C_5 of the characters of ρ_1 and sign equal to the trivial character, the character of ρ_ω is the sum of the characters of τ_ω and τ_{ω^4}, in a similar way ρ_{ω^2} is the direct sum of τ_{ω^2} and τ_{ω^3}.

7. The quaternion group Q is the group $Q = \langle i, j, k \mid i^2 = j^2 = k^2 = -1, ijk = -1\rangle$

   (a) Find a subgroup of $\mathrm{GL}_2(\mathbb{C})$ isomorphic to Q and determine the order of Q.

   (b) Determine the conjugacy classes of Q.

   (c) Show that any subgroup of Q is normal.

   (d) Determine the character table of Q.

   Solution:

   (a) Let us consider the elements of $\mathrm{GL}_2(\mathbb{C})$ $I = \left(\begin{smallmatrix} i & 0 \\ 0 & -i \end{smallmatrix}\right)$, $J = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$, $K = \left(\begin{smallmatrix} 0 & i \\ i & 0 \end{smallmatrix}\right)$. We have $I^2 = -\mathrm{id}_\mathbb{C}$, $J^2 = -\mathrm{id}_\mathbb{C}$, $K^2 = -\mathrm{id}_\mathbb{C}$, moreover

   $$IJK = \left(\begin{smallmatrix} i & 0 \\ 0 & -i \end{smallmatrix}\right)\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)\left(\begin{smallmatrix} 0 & i \\ i & 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 & i \\ i & 0 \end{smallmatrix}\right)\left(\begin{smallmatrix} 0 & i \\ i & 0 \end{smallmatrix}\right) = -\mathrm{id}_\mathbb{C}.$$

   In particular the subgroup Q can be realized as a subgroup of $\mathrm{GL}_2(\mathbb{C})$ and has eight elements: $\{\pm\mathrm{id}_\mathbb{C}, \pm I, \pm J, \pm K\}$.

   (b) We will now identify Q with the subgroup of $\mathrm{GL}_2(\mathbb{C})$ we just defined, to make explicit computations. Since the matrices $\pm Id$ commute with every matrix in $\mathrm{GL}_2(\mathbb{C})$, in particular they commute with the elements in Q, hence they are in the center of Q. Moreover, from the fact that $I^2 = -1$ we get that $I^{-1} = -I$. We can now compute the relation

   $$IJI^{-1} = -\left(\begin{smallmatrix} i & 0 \\ 0 & -i \end{smallmatrix}\right)\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)\left(\begin{smallmatrix} i & 0 \\ 0 & -i \end{smallmatrix}\right) = -J.$$

Analogously one gets that $KJK^{-1} = -J$, in particular the conjugacy class of J contains the two elements $\pm J$. In the same way one checks that the conjugacy class of I contains $\pm I$ and the conjugacy class of K contains $\pm K$.

(c) The computation above shows that the subgroup generated by I is isomorphic to $\mathbb{Z}/4\mathbb{Z}$ and is normal and the same is true for $\langle J \rangle$ and $\langle K \rangle$. The only other non trivial subgroup is the center $\{\pm 1\}$ of Q and clearly the center is normal.

(d) In order to compute the character table we need to find 5 irreducible representations of Q, since there are 5 conjugacy classes. Of course there is the trivial representation, which we will denote by ρ_1. Moreover, we saw above that the subgroup generated by I is normal. Since the quotient Q/I is $\mathbb{Z}/2\mathbb{Z}$ we get a representation ρ_i obtained by composing the sign representation of $\mathbb{Z}/2\mathbb{Z}$ with the quotient map. In the same way (quotienting the subgroup generated by J and K respectively) one gets the representations ρ_j and ρ_k. All the representations ρ_1, ρ_i, ρ_j and ρ_k are one-dimensional, hence irreducible. Moreover, by computing the characters, it is easy to see that they are not isomorphic, since the characters are different. We have realized Q as a subgroup of $\mathrm{GL}_2(\mathbb{C})$. This gives a two dimensional representation ρ of Q which is irreducible since the eigenspaces of I and J are distinct. We can sum up the results we just obtained in the character table for Q:

Q	$\{1\}$	$\{-1\}$	$\{\pm i\}$	$\{\pm j\}$	$\{\pm k\}$
ρ_1	1	1	1	1	1
ρ_i	1	1	1	-1	-1
ρ_j	1	1	-1	1	-1
ρ_k	1	1	-1	-1	1
ρ	2	-2	0	0	0

No hand-in. *Enjoy your semester break!*