1. Let $K$ be a splitting field of $x^7 - 5$ over $\mathbb{Q}$. Prove that $K$ contains a primitive $7^{th}$ root of unity. Determine the degree of $K/\mathbb{Q}$.

2. Let $K = \mathbb{Q}(\sqrt[5]{3}, \zeta)$, where $\zeta$ is a primitive $15^{th}$ root of unity.

   (a) Prove that $K$ is Galois over $\mathbb{Q}$ and find the order the Galois group $G$.

   (b) Show that $\mathrm{Gal}(K/\mathbb{Q}(\zeta))$ is a normal subgroup of $G$.

3. Let $G$ be a finite group. Does there always exist a finite dimensional extension of fields $E/F$ which is Galois with Galois group isomorphic to $G$?

4. Let $\mathbb{Q}(\zeta)/\mathbb{Q}$ be the extension $\mathbb{Q}(\zeta) \subset \mathbb{C}$ of the rational numbers determined by

$$\zeta = \exp(2\pi i/29)\,.$$

   Let $\alpha \in \mathbb{Q}(\zeta)$ be a number which is constructible over $\mathbb{Q}$. What are the possible values of

$$\dim_{\mathbb{Q}} \mathbb{Q}(\alpha)/\mathbb{Q}\,?$$

5. Let $E$ be the splitting field of the polynomial $x^4 - 2x^2 - 3$ over $\mathbb{Q}$. What is the degree of $E/\mathbb{Q}$? Find all proper subextensions of $E/\mathbb{Q}$.

6. Solve the equation $x^3 - 6x - 9 = 0$ using Cardano's formula.

7. Let $\mathbb{F}_2$ be the field with 2 elements.

   (a) Factor the polynomial

$$f(x) = x^4 + x^2 + x + 1 \in \mathbb{F}_2[x]$$

   into irreducible factors in the ring $\mathbb{F}_2[x]$.

   (b) Let $E$ be the splitting field of $f$ over $\mathbb{F}_2$. How many elements does $E$ have?

8. Is the regular 272-gon constructible by straightedge and compass?

9. Let $\mathbb{C}(e_1, e_2, e_3) \subset \mathbb{C}(x_1, x_2, x_3)$ be the subfield of symmetric functions where

$$e_1 = x_1 + x_2 + x_3\,, \quad e_2 = x_1x_2 + x_1x_3 + x_2x_3\,, \quad e_3 = x_1x_2x_3\,.$$

   What is the degree of the extension of $\mathbb{C}(e_1, e_2, e_3)$ generated by the element

$$x_1 + 2x_2 + 3x_3 \in \mathbb{C}(x_1, x_2, x_3)\,?$$

10. Let $\mathbb{C}(e_1, e_2, e_3) \subset \mathbb{C}(x_1, x_2, x_3)$ be the subfield of symmetric functions where

$$e_1 = x_1 + x_2 + x_3\,, \quad e_2 = x_1x_2 + x_1x_3 + x_2x_3\,, \quad e_3 = x_1x_2x_3\,.$$

   What is the degree of the extension of $\mathbb{C}(e_1, e_2, e_3)$ generated by the element

$$x_1^2x_2 + x_2^2x_3 + x_3^2x_1 \in \mathbb{C}(x_1, x_2, x_3)\,?$$

11. Let $\mathbb{C}(e_1, e_2, e_3) \subset \mathbb{C}(x_1, x_2, x_3)$ be the subfield of symmetric functions where

$$e_1 = x_1 + x_2 + x_3\,, \quad e_2 = x_1x_2 + x_1x_3 + x_2x_3\,, \quad e_3 = x_1x_2x_3\,.$$

What is the minimal polynomial over $\mathbb{C}(e_1, e_2, e_3)$ of the element

$$x_1 + x_2 \in \mathbb{C}(x_1, x_2, x_3)\,?$$

12. Factorize $x^8 - 1$ and $x^9 - 1$ into irreducible factors in the ring $\mathbb{Q}[x]$.

13. Does there exist an irreducible polynomial $f(x) \in \mathbb{Q}[x]$ of degree 4 with Galois group over $\mathbb{Q}$ isomorphic to $S_3 \times S_3$? Does there exist an irreducible polynomial $g(x) \in \mathbb{Q}[x]$ of degree 5 with Galois group over $\mathbb{Q}$ isomorphic to $S_3 \times S_3$?

14. What is the automorphism group of the field $\mathbb{Q}$? What is the automorphism group of the field $\mathbb{R}$? Answer with proof.

15. Let $\mathbb{F}_{243}$ be the field with 243 elements.

    (a) Let $\tau : \mathbb{F}_{243} \to \mathbb{F}_{243}$ be defined by $\tau(x) = x^{80}$. Is $\tau$ an automorphism of $\mathbb{F}_{243}$? If so, calculate the number of elements of the fixed field of $\tau$.

    (b) Let $\sigma : \mathbb{F}_{243} \to \mathbb{F}_{243}$ be defined by $\tau(x) = x^{81}$. Is $\sigma$ an automorphism of $\mathbb{F}_{243}$? If so, calculate the number of elements of the fixed field of $\sigma$.

16. Let $\mathbb{F}_2$ be the field with 2 elements. Consider the polynomial

$$f(x) = x^3 + x^2 + 1 \in \mathbb{F}_2[x]\,.$$

Is $f(x)$ irreducible in $\mathbb{F}_2[x]$? What is the Galois group $f(x)$ over $\mathbb{F}_2$?

17. Consider the real number

$$\alpha = \sqrt{17} + \frac{2}{\sqrt{17}}\,.$$

Is $\alpha$ algebraic over $\mathbb{Q}$? If so, find the minimal polynomial of $\alpha$ over $\mathbb{Q}$.

18. What is the Galois group of the polynomial $x^{17} - 1$ over $\mathbb{Q}$? Does there exist a subextension of the splitting field over $\mathbb{Q}$ of dimension 4?

19. Let $G$ be the group of rotational symmetries of a solid tetrahedron. What is the order of $G$? Is $G$ a solvable group?

20. Let $f(x) = x^3 + 2x^2 + 3x + 4$. Let $r_1, r_2, r_3 \in \mathbb{C}$ be the three roots of $f(x)$. Calculate

$$r_1^3 + r_2^3 + r_3^3\,.$$

21. Let $E$ be a Galois extension of $F$ with Galois group $S_3$.

    (a) What is the degree of the extension $E/F$?

    (b) How many proper subextensions of $E/F$ are there?

    (c) How many of the proper subextensions of $E/F$ are Galois extensions of $F$?

22. Let $f(x) \in \mathbb{Q}[x]$ be a polynomial of degree 4 with distinct roots

$$r_1, r_2, r_3, r_4 \in \mathbb{C}$$

which satisfy the equation

$$\prod_{i<j}(r_i - r_j) = 9\,.$$

Can the Galois group of $f$ over $\mathbb{Q}$ be $S_4$? Answer with proof.

23. Let $\omega = \exp(2\pi i/3)$. Consider the complex number

$$\alpha = \omega + 2\omega^{-1}\,.$$

Is $\alpha$ algebraic over $\mathbb{Q}$? If so, find the minimal polynomial of $\alpha$ over $\mathbb{Q}$.

24. Let $f(x) \in \mathbb{Q}[x]$ be an irreducible polynomial of degree 17. Let $\alpha \in \mathbb{C}$ satisfy

$$f(\alpha) = 0\,.$$

(a) Can $\alpha$ be a rational number?
(b) What is the degree of the extension $\mathbb{Q}(\alpha)/\mathbb{Q}$?
(c) What is the degree of the extension $\mathbb{Q}(\alpha^2 + 17)/\mathbb{Q}$?

25. Let $E/\mathbb{Q}$ be a finite dimensional field extension of $\mathbb{Q}$.

(a) Suppose $\dim E/\mathbb{Q} = 6$ and

$$\mathbb{Q} \subset F_1 \subset E\,, \quad \mathbb{Q} \subset F_2 \subset E$$

are subextensions with

$$\dim F_1/E = 2\,, \quad \dim F_2/E = 3\,.$$

Let $z_1 \in F_1$ and $z_2 \in F_2$ with $z_1, z_2 \notin \mathbb{Q}$. Must there exists $\lambda_1, \lambda_2 \in \mathbb{Q}$ such that

$$\mathbb{Q}(\lambda_1 z_1 + \lambda_2 z_2) = E\,?$$

(b) Suppose $\dim E/\mathbb{Q} = 12$ and

$$\mathbb{Q} \subset F_1 \subset E\,, \quad \mathbb{Q} \subset F_2 \subset E$$

are subextensions with

$$\dim F_1/E = 4\,, \quad \dim F_2/E = 3\,.$$

Let $z_1 \in F_1$ and $z_2 \in F_2$ with $z_1, z_2 \notin \mathbb{Q}$. Must there exists $\lambda_1, \lambda_2 \in \mathbb{Q}$ such that

$$\mathbb{Q}(\lambda_1 z_1 + \lambda_2 z_2) = E\,?$$