# Constructions w/ straightedge and compass.

**Goal.** Characterise the set of complex numbers which can be constructed from a given set using straightedge and compass.

## I. Basic operations and constructions.

Let $E$ denote the euclidean plane.

### Permitted operations.

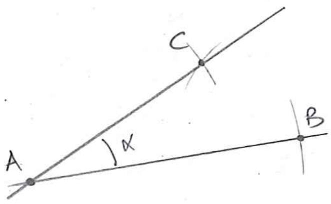
(1) create a straight line between two existing points;

(2) take the distance between two existing points and create a circle with this radius and centre any existing point;

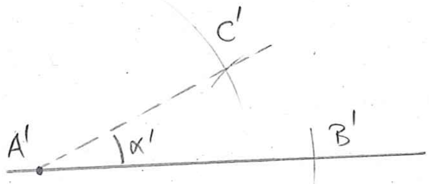
(3) create the one or two points in the intersection of any two lines as in (1) or (2) which intersect.

### Basic constructions.

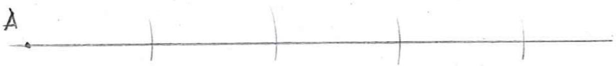
(a) transfer a length onto a line segment




(b) transfer an angle

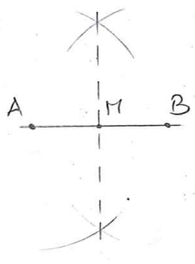



(c) construct arbitrarily large lengths

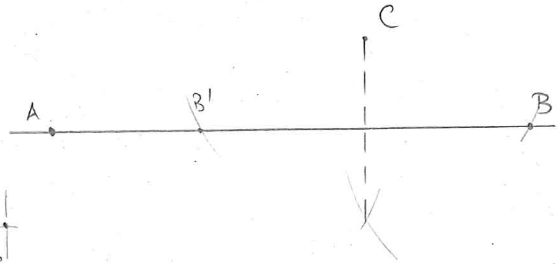



(d) find the midpoint of a line segment

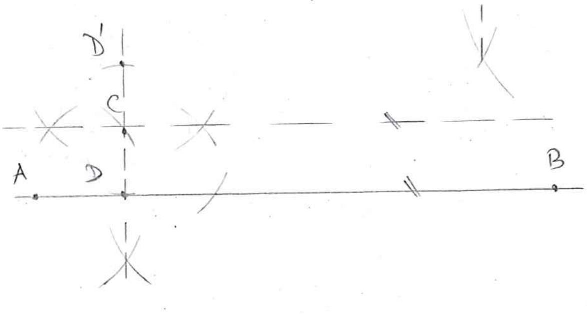
(e) construct the perpendicular bisector




(f) construct a perpendicular line from a point to a line




(g) construct a line through a point parallel to a given line



# II. Constructible numbers.

Let $S \subset E$ be a set of points and let $K_S$ denote the set of all points which can be constructed from $S$ using (1) - (3).

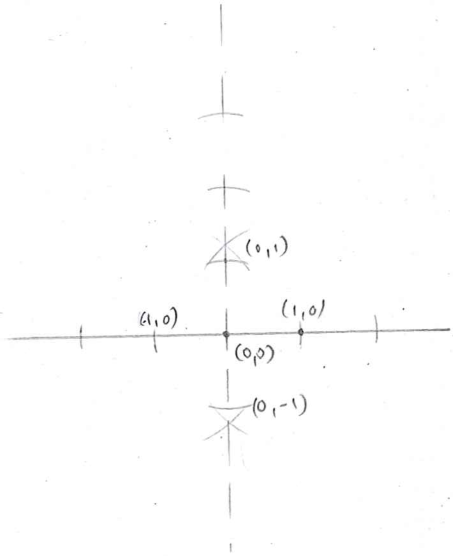
Assume $\# S \geq 2$ (otherwise $K_S = S$)

**Def 1.** For $A, B, C \in K_S$, the (euclidean) distance $d(A,B) \in \mathbb{R}_{\geq 0}$ is called an <u>S-constructible length</u>, and the angle $\measuredangle ABC$ is called an <u>S-constructible angle</u>.

Identify $E \cong \mathbb{R}^2$ in such a way that $(0,0), (1,0) \in S$.

**Fact 2.** $(0,n), (n,0) \in K_S$ for all $n \in \mathbb{Z}$:

- transfer length $d((0,0),(1,0))$ to get $(-1,0)$ (construction (a))
- construct perp bisec (constr. (e)) and transfer length again (constr. (a)) to get $(0,1)$ and $(0,-1)$
- repeat constr. (a) $|n|$ times along axes in either direction to get $(0,n)$ and $(n,0)$.




Identify $\mathbb{R}^2 \cong \mathbb{C}$ as usual via $(x,y) \leftrightarrow x+iy$ $\rightsquigarrow$ wlog $E = \mathbb{C}$ and $0, 1 \in S$

$\overset{\text{Fact 2}}{\Longrightarrow} \mathbb{Z}, i\mathbb{Z} \in K_S$.

**Theorem 3.** $K_S$ is the smallest subfield of $\mathbb{C}$ with the properties

(i) $S \subset K_S$

(ii) $\forall z \in K_S : \bar{z} \in K_S$

(iii) $\forall z \in \mathbb{C} : z^2 \in K_S \Rightarrow z \in K_S$

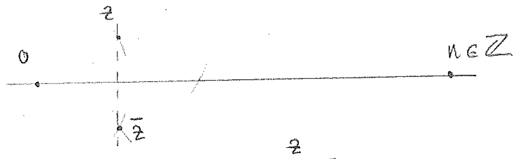
i.e. $K_S$ is closed under complex conjugation and extraction of square roots.

**Def 4.** We call $z \in K_S$ an <u>S-constructible number</u>.

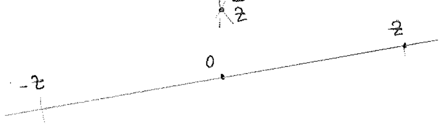
If $S = \{0,1\}$, then $z \in K_S$ is a constructible number.

Proof of Thm 3. (i) is clear.

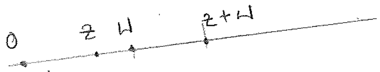
1. $z \in K_S \Rightarrow \bar{z} \in K_S$ : use constr. (f) + (a)




2. $z \in K_S \Rightarrow -z \in K_S$ : use constr (a)

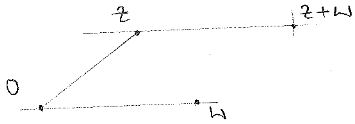



3. $z, w \in K_S \Rightarrow z+w \in K_S$ :
   - $z=0$ or $w=0$ is clear.
   - $z, w$ collinear :

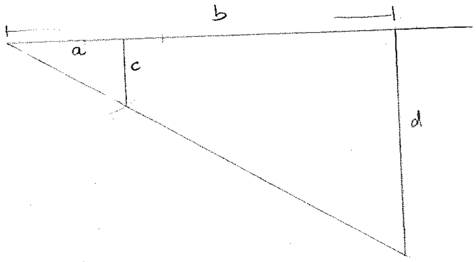



construction (a)

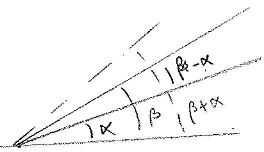
   - otherwise : construct parallelogram (constr. (g))



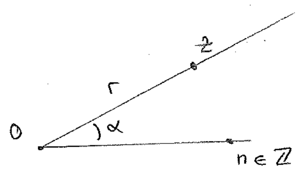

4. if $a, b, c > 0$ are $S$-constructible lengths, then so is $bc/a$ : use basic constr's to obtain $c \perp a$ ; $d \perp b \supset a$ ⤳ basic proportionality thm implies $\frac{a}{c} = \frac{b}{d} \Rightarrow d = \frac{bc}{a}$ is constructible.




5. if $\alpha, \beta \in \mathbb{R}$ are $S$-constructible angles, then so are $\alpha \pm \beta$ :

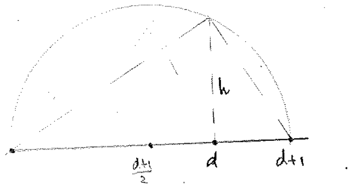



6. Write $z = re^{i\alpha}$ with $r \in \mathbb{R}_{>0}$, $\alpha \in \mathbb{R}$. Then $z \in K_S \Leftrightarrow r$ and $\alpha$ are $S$-constructible.

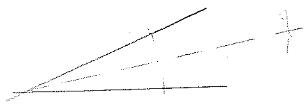



7. $z, w \in K_S \Rightarrow zw \in K_S$ and if $w \neq 0$, then $z/w \in K_S$ : Write $z = re^{i\alpha}$, $w = se^{i\beta}$. Then $zw = rse^{i(\alpha+\beta)}$ is constructible if $z$ and $w$ are (combine 4, 5 and 6) w/ $b=r, c=s, a=1$ and $b=r, c=1, a=s$

8. If $d > 0$ is an $S$-constructible length, then so is $\sqrt{d}$ : use basic constr's and Thales thm to get a right triangle. Then geometric mean thm says $h = \sqrt{1 \cdot d} = \sqrt{d}$.




9. If $\alpha$ is an $S$-constructible angle, then so is $\frac{\alpha}{2}$ :



10. $z^2 \in K_S \Rightarrow z \in K_S$ : Write $z^2 = re^{i\alpha}$. Combine 2, 6, 8, 9 to get $z = \pm\sqrt{r}e^{i\alpha/2} \in K_S$

It follows that $K_S$ is a subfield of $\mathbb{C}$ satisfying (i) - (iii).

Remains to show : $K_S$ is the <u>smallest</u> such field.

**Facts 5.** • the $S$-constructible lengths are precisely the ones in $K_S \cap \mathbb{R}_{\geq 0}$

• for all $x, y \in \mathbb{R}$ : $z := x + iy \in K_S \Leftrightarrow x, y \in K_S \cap \mathbb{R}$

• the line passing through two points $a+ib \neq c+id$ is given by

$$\left\{ x+iy \in \mathbb{C} \;\middle|\; x,y \in \mathbb{R} \text{ and } \det\begin{pmatrix} x & a & c \\ y & b & d \\ 1 & 1 & 1 \end{pmatrix} = 0 \right\}$$

• the circle with radius $r$ and midpoint $a+ib$ is

$$\left\{ x+iy \in \mathbb{C} \;\middle|\; x,y \in \mathbb{R} \text{ and } (x-a)^2 + (y-b)^2 = r^2 \right\}$$

$\Rightarrow$ • the intersection of two lines is obtained by elementary arithmetic, ie by using only the 4 basic arithmetic operations $+, -, \cdot, /$ (two linear equations for $x$ and $y$)

• the intersection of a line and a circle or of two circles is obtained by elementary arithmetic and extraction of square roots:
line $\cap$ circle: eliminate one variable from line equ $\rightsquigarrow$ quadratic equ
circle $\cap$ circle: difference of the two circ equs is linear.

**Consequence 6.** All $z \in K_S$ are obtained from $S$ by elementary arithmetic and extraction of square roots and complex conjugation.
$\Rightarrow K_S$ is the smallest such subfield.

$\square$ Thm 3.

Set $K_0 := \mathbb{Q}(S \cup \{\bar{s} \mid s \in S\})$. Note: $S = \{0,1\} \Rightarrow K_0 = \mathbb{Q}$.

**Corollary 7.** $z \in \mathbb{C}$ is an $S$-constructible number if and only if there exists $n \geq 0$ and a tower of field exts $K_0 \subset K_1 \subset \ldots \subset K_n$ st $z \in K_n$ and $[K_j / K_{j-1}] = 2$ for all $1 \leq j \leq n$. (follows immediately from Cons. 6)

**Corollary 8.** If $z \in \mathbb{C}$ is $S$-constructible, then $[K_0(z)/K_0] = 2^m$ for some $m \geq 0$.

*Proof.* $K_0(z)$ is an intermediate field of $K_n/K_0$, so $[K_0(z)/K_0]$ divides $[K_n/K_0] = 2^n$. $\square$

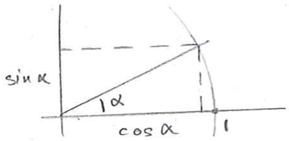
**Caution.** Not every extension $L/K_0$ of degree $2^m$ for $m \geq 2$ can be embedded into such a $K_n$. $\rightarrow$ see exercise in Assignment 21.

# III. Applications to geometric constructions.

**Proposition 9.** For $\alpha \in \mathbb{R}$ tfae:

(i) $\alpha$ is $S$-constructible (ii) $\cos\alpha \in K_S \cap \mathbb{R}$ (iii) $\sin\alpha \in K_S \cap \mathbb{R}$

Proof.




**Example 10.** The regular 5-gon can be constructed.

Proof. Set $\alpha := \frac{2\pi}{5}$ and $e^{i\alpha} = \cos\alpha + i\sin\alpha =: x + iy$. Then

$$1 = (x+iy)^5 = x^5 + \cancel{5ix^4y} - 10x^3y^2 - \cancel{10ix^2y^3} + 5xy^4 + \cancel{iy^5} = x^5 - 10x^3y^2 + 5xy^4$$

$(\mathrm{Im}(LHS) = 0)$

$$\overset{y^2 = \sin^2 = 1-\cos^2 = 1-x^2}{=} x^5 - 10x^3(1-x^2) + 5x(1-x^2)^2 = 16x^5 - 20x^3 + 5x$$

(Note: $\cos 5\alpha = 16\cos^5\alpha - 20\cos^3\alpha + 5\cos\alpha$)

$\Rightarrow \cos\alpha$ is a root of $f(X) := 16X^5 - 20X^3 + 5X - 1 = (X-1)(4X^2 + 2X - 1)^2$

$\cos\alpha > 0 \Rightarrow \cos\alpha = \frac{1}{4}(\sqrt{5} - 1)$, which is constructible.

**Theorem 11.** (Gauss-Wantzel, w/o proof).

A regular $n$-gon can be constructed w/ straightedge and compass if and only if $n$ is the product of a power of 2 and a (possibly empty) product of distinct Fermat primes (ie primes of the form $2^{2^n} + 1$)

$\rightsquigarrow$ only 5 known Fermat primes $\Rightarrow$ only 31 known odd-sided constructible $n$-gons!

**Proposition 12.** The following constructions are impossible using only straightedge and compass and finitely many steps:

(1) doubling the cube: given a cube of volume $v$ and side length $s$, construct a cube of volume $2v$.

(2) trisecting the angle: given an arbitrary angle $\alpha$, construct $\alpha/3$.

(3) squaring the circle: construct a square with the same area as a given circle.

Proof. (1) WLOG $s=1 \rightsquigarrow K_0 = \mathbb{Q} \Rightarrow$ need to construct $\sqrt[3]{2}$. But this is a root of $X^3-2$, which is irred $/\mathbb{Q}$ by Eisenstein criterion at $p=2$.

Thus $[\mathbb{Q}(\sqrt[3]{2})/\mathbb{Q}]=3$ is not a power of 2, and $\sqrt[3]{2} \notin K_S$.

(2) Let $\alpha = \frac{\pi}{3}$. Then $\cos\alpha = \frac{1}{2} \in \mathbb{Q} \subset K_{\{0,1\}} \cap \mathbb{R}$, so $\alpha$ is constructible. But, using trigonometric identities, we have $\frac{1}{2} = \cos\alpha = \cos(3\cdot\alpha/3) = 4\cos^3(\alpha/3) - 3\cos(\alpha/3)$. $(*)$

Set $u := 2\cos(\alpha/3)$. Then $(*) \Leftrightarrow u^3 - 3u = 1$. So $\alpha$ is constructible if and only if the polynomial $X^3-3X-1$ is reducible over $\mathbb{Q}$. This is true if and only if $(X+1)^3 - 3(X+1) - 1 = X^3+3X^2-3$ is. But the latter is irreducible by Eisenstein at $p=3$. So $[\mathbb{Q}(\cos(\alpha/3))/\mathbb{Q}]=3$ and $\alpha/3$ not constructible.

(3) Take $S=\{0,1\}$, $r=1 \rightsquigarrow$ area $=\pi$. Then we need to construct side length $\sqrt{\pi}$. But we have $[\mathbb{Q}(\pi)/\mathbb{Q}(\sqrt{\pi})]=2$ and the highly nontrivial fact that $[\mathbb{Q}(\pi)/\mathbb{Q}]=\infty$. Thus we deduce from $\heartsuit: [\mathbb{Q}(\pi)/\mathbb{Q}] = [\mathbb{Q}(\pi)/\mathbb{Q}(\sqrt{\pi})]\cdot[\mathbb{Q}(\sqrt{\pi})/\mathbb{Q}]$

that $[\mathbb{Q}(\sqrt{\pi})/\mathbb{Q}]=\infty$ and so $\sqrt{\pi}$ not constructible. $\square$