Multiple Choice.

1. Let G be a group and H a subgroup of G. Which of the following statements are *always* true in this case?
   (a) For G finite, the index of H in G divides the order of G.
   (b) G contains a normal subgroup whose index is a prime number.
   (c) If H is abelian and normal in G and G/H is abelian, then G is abelian.
   (d) If the index of H in G is two, the subgroup H is a normal subgroup of G.
   (e) For any two simple groups G_1, G_2, their direct sum $G_1 \oplus G_2$ is also simple.

   Recall: A group G is called *simple* if the only normal subgroups of G are $\{1\}$ and G.

2. Let G be a group acting on a set X and let H be a subgroup of G. Which of the following statements are *always* true in this case?
   (a) For any field k, the action of the group $\mathrm{SL}_n(k)$ on k^n where a matrix $A \in \mathrm{SL}_n(k)$ acts on a vector $v \in k^n$ by multiplication from the left is a faithful action.
   (b) If there exists $x \in X$ with $\mathrm{Stab}_G(x) = \{1\}$ then the action of G on X is faithful.
   (c) If the action of H on X is transitive, so is the action of G on X.
   (d) There is a map from the set of G-orbits to the set of H-orbits which sends Gx to Hx.
   (e) The only action of the group $G = \mathbb{Z}/2\mathbb{Z}$ on the set $X = \{1, 2, 3\}$ is the trivial action $gx = x$ for $g \in G, x \in X$.

   Recall: An action of G on X is called *faithful* if the only element $g \in G$ satisfying $gx = x$ for all $x \in X$ is the neutral element $g = 1$.

3. (a) The group S_n has a subgroup of order m for each $m = 1, 2, \ldots, n$.
   (b) The 6-cycle $\sigma = (1\,2\,3\,4\,5\,6) \in S_6$ has a decomposition into a product $\sigma = \tau_1\tau_2\cdots\tau_m$ of transpositions $\tau_1, \tau_2, \ldots, \tau_m \in S_6$ and for each such decomposition the number m of transpositions is odd.
   (c) The 6-cycle $\sigma = (1\,2\,3\,4\,5\,6) \in S_6$ cannot be written as the product of 3 transpositions.
   (d) The group A_5 is the only simple subgroup of S_5.
   (e) The subgroup $G = \langle (1\,2), (3\,4\,5) \rangle \subset S_5$ is isomorphic to S_6.

4. Let A be a commutative ring and I an ideal of A. Which of the following statements are then *always* true?

   (a) For all $f, g \in A[X]$ of degree 3, we have that $f + g$ also has degree 3.

   (b) If I is a prime ideal, then for $a, b \in A/I$ with $a \neq 0, b \neq 0$ we have $ab \neq 0$.

   (c) For any PID A, also $A[X]$ is a PID.

   (d) The ideal $I = (X)$ in $A[X]$ is maximal.

   (e) If $I \subsetneq A$ is a proper ideal, then there exists a prime ideal of A containing I.

   Recall: We say that a polynomial $f \in A[X]$ has degree m if f has the form $f(X) = a_m X^m + a_{m-1}X^{m-1} + \ldots + a_1 X + a_0$ with $a_m \neq 0$.

5. Let A and B be commutative rings and $f : A \to B$ a ring homomorphism. Let I be an ideal of A and denote by $p : A \to A/I$ the usual projection. Which of the following statements are then *always* true?

   (a) There exists a ring homomorphism $g : A/I \to B$ satisfying $f = g \circ p$ if and only if f vanishes on I.

   (b) If $f \neq 0$ and B is an integral domain, then $A/\ker(f)$ is an integral domain.

   (c) If f is surjective, then B is isomorphic to a subring of A.

   (d) For every $b_1, b_2 \in B$ there exists a unique ring homomorphism $h : A[X, Y] \to B$ sending $X \mapsto b_1, Y \mapsto b_2$ and such that $h|_A = f$, where we use the natural inclusion $A \subset A[X, Y]$.

   (e) If $J \subset B$ is a maximal ideal and $f : A \to B$ is surjective, then $f^{-1}(J)$ is a maximal ideal in A.

6. Let $R = \mathbb{Z}/30\mathbb{Z}$.

   (a) The natural map $R \to (\mathbb{Z}/2\mathbb{Z}) \oplus (\mathbb{Z}/3\mathbb{Z}) \oplus (\mathbb{Z}/5\mathbb{Z})$ sending $\bar{a}$ to $(\bar{a}, \bar{a}, \bar{a})$ is an isomorphism.

   (b) There exist exactly 3 elements of order 3 in the group $(R, +, 0)$.

   (c) The ideal $(\bar{5}) \subset R$ is maximal.

   (d) There exist exactly 3 distinct ring homomorphisms $R \to \mathbb{Z}/3\mathbb{Z}$.

   (e) Every ideal in R is a principal ideal.

7. (a) Every torsion-free $\mathbb{Z}$-module is free.

   (b) Every free $\mathbb{Z}$-module A with $A \neq 0$ contains $\mathbb{Z}$ as a submodule.

   (c) There are, up to isomorphism, 4 different abelian groups of 8 elements.

   (d) If A is a finitely generated $\mathbb{Z}$-module in which each element has finite order, then A is finite.

   (e) If A is a finitely generated $\mathbb{Z}$-module in which each nonzero element has infinite order, then A is free.

8. Let L/K be a field extension.
   (a) If $\dim_K L = 17$ then the field extension is algebraic.
   (b) If $f \in K[T]$ has two distinct roots in L, then f cannot be irreducible in $K[T]$.
   (c) If $\alpha \in L \setminus \{0\}$ is transcendental over K, then also $1/\alpha$ is transcendental.
   (d) If $\alpha \in L$ is algebraic and L/K is of finite degree, then $\deg(\operatorname{irr}(\alpha; k))$ divides $[L : K]$.
   (e) For $\alpha \in L$, the field $K(\alpha) \subset L$ is equal to the K-vector subspace of L spanned by $1, \alpha, \alpha^2, \ldots$.

   Recall: If $\alpha \in L$ is algebraic over K, its minimal polynomial $\operatorname{irr}(\alpha; k)$ is the unique monic polynomial in $K[X]$ of minimal degree among all nonzero polynomials in $K[X]$ that vanish on α.

9. (a) For every finite field $\mathbb{F}$, the characteristic $\operatorname{char}(\mathbb{F})$ divides the number of elements $|\mathbb{F}|$ of $\mathbb{F}$.
   (b) If K is an extension of degree 2 of the finite field $\mathbb{F}_4$, then K is isomorphic to $\mathbb{F}_8$.
   (c) For each element $a \in \mathbb{F}_q \setminus \{0\}$ one has $a^q = 1$.
   (d) The polynomial $X^{121} - 1$ has exactly 10 distinct roots in $\mathbb{F}_{11}$.
   (e) If E is a finite field and F/E is an algebraic extension, then F is a finite field.

   Recall: For a field K the characteristic $\operatorname{char}(K)$ is defined to be $\operatorname{char}(K) = 0$ if there exists no integer $n \in \mathbb{Z}_{>0}$ with $n \cdot 1_K = 0_K$. Otherwise, $\operatorname{char}(K)$ is defined to be the minimal positive integer $n \in \mathbb{Z}_{>0}$ with $n \cdot 1_K = 0_K$.

10. (a) The splitting field in $\mathbb{C}$ of the $\mathbb{Q}$-polynomial $X^2 - 2$ is $\mathbb{Q}(\sqrt{2})$.
    (b) The splitting field in $\mathbb{C}$ of the $\mathbb{Q}$-polynomial $X^2 + 5$ is $\mathbb{Q}(\sqrt{5}, i)$.
    (c) The splitting field in $\mathbb{C}$ of the $\mathbb{Q}$-polynomial $X^{17} - 1$ is isomorphic to $\mathbb{Q}[X]/(X^{17} - 1)$.
    (d) For any $\alpha \in \mathbb{C}$ there exists a $\mathbb{Q}$-polynomial $f(X) \in \mathbb{Q}[X]$ such that $\mathbb{Q}(\alpha)$ is the splitting field in $\mathbb{C}$ of $f(X)$.
    (e) The field $\mathbb{Q}(\sqrt{7}, i)$ has degree 4 over $\mathbb{Q}$.