Course Summary
Topology 2019

Prof. Alessandro Sisto

Lecture 1, 18.02.2019

Introduction, motivation and outline of the course.

Lecture 2, 20.02.2019

- Definition open set in $\mathbb{R}$ and first examples.
- Definition of continuity of $f : \mathbb{R} \longrightarrow \mathbb{R}$.
- Proposition: A function $f : \mathbb{R} \longrightarrow \mathbb{R}$ is continuous (according to the usual ϵ, δ definition) if and only if for every open set $U \subseteq \mathbb{R}$, the inverse image $f^{-1}(U)$ is open in $\mathbb{R}$.
- Definitions of topological space, open set, topology, trivial topology, discrete topology.
- Proposition: The collection of open subsets of $\mathbb{R}$ is a topology.
- Definition of the standard topology on $\mathbb{R}$.
- Examples of some "alternatives" topologies on $\mathbb{R}$.
- Definitions of coarser and finer topology, closed set.
- Let X a set, and S a family of subsets of X such that
  - $\emptyset, X \in S$
  - Given $s_1, s_2, \ldots, s_n \in S$, then $\bigcup_{i=1}^{n} s_i \in S$. (Finite union property)
  - Given $\{s_i\}_{i \in I}$ such that $s_i \in S$ for all $i \in I$, then $\bigcap_{i \in I} s_i \in S$. (Intersection property).

  Then the family $\tau = \{X \setminus s : s \in S\}$ is a topology on X.

Lecture 3, 25.02.2019

- Lemma: Given a topological space X, and $A \subseteq X$, for all $x \in X$, exactly one of the following conditions holds:
  - There exists an open U such that $x \in U$ and $U \subseteq A$;
  - There exists an open U such that $x \in U$ and $U \subseteq X \setminus A$;
  - For all U open with $x \in U$, we have that $U \cap A \neq \emptyset$ and $U \cap (X \setminus A) \neq \emptyset$.
- Interior, boundary and closure of a set.
- Lemma: $\text{int}(A) \subseteq A \subseteq \overline{A}$.
- Basic examples of interiors and closures.
- Proposition: Let X be a topological space, and $A \subseteq X$
  - $\text{int}(A)$ is open;
  - $\overline{A}$ is closed;
  - A is open $\iff A = \text{int}(A)$;
  - A is closed $\iff A = \overline{A}$.
- Exercises: Show that ∂A is always closed, that $\text{int}(A)$ is the maximal open set include in A, and that $\overline{A}$ in the minimal closed set which includes A.

Lecture 4, 27.02.2019

- Basis of a topology.
- Remark: If $\mathbb{B}$ is a basis, then the following two properties hold:
  1. For all $x \in X$, the element x is contained in some element of the basis;
  2. Let $B_1, B_2 \in \mathbb{B}$, and let $x \in B_1 \cap B_2$, then there exists a basis element B_3 such that $x \in B_3$ and $B_3 \subseteq B_1 \cap B_2$.
- Proposition (Characterisation of a basis): Let X be a topological space and $\mathbb{B} \subseteq \mathcal{P}(X)$. If X satisfies the properties 1 and 2 above, then $\mathbb{B}$ is a basis.
- Product topology.
- Metric space, and standard basis for a metric space.
- Basic examples of metric spaces.

- Subspace topology.
- Homeomorphism, and basic examples.
- Proposition: The following topologies on $\mathbb{R}^n$ coincide
  - Product topology;
  - the metric topology with regard to euclidean distance.

Lecture 5, 04.03.2019

- Disconnected and connected spaces, first examples.
- Proposition: The intervals in $\mathbb{R}$ are connected.
- Path-connectedness
- Proposition: Let X be a topological space. If X is path connected, then it is connected.
- Proposition: Let $f : X \longrightarrow Y$ be a continuous surjective map. If X is path connected, then so is Y.

Lecture 6, 06.03.2019

- Cut point, definition and first examples
- The number of cut points is invariant under homeomorphism
- The *topologists' sine curve*, as an example of connected but not path-connected space
- Definitions of inverse path, connected, disconnected, path-connected subspaces
- A topological space is the disjoint union of its path-connected components
- If $A \subseteq X$ is a path-connected subspace, then it is contained in a path-connected component of X
- Denote by $P(x)$ the path-connected component of $x \in X$, and let $f : X \longrightarrow Y$ be continuous, then $f(P(x)) \subseteq P(f(x))$
- If $A \subseteq X$ is connected, then $\overline{A}$ is connected. This is not true for the interior
- Let $A \subseteq X$ open and closed, and $C \subseteq X$ connected. If $C \cap A \neq \emptyset$, then $C \subseteq A$

- Let $\{C_\alpha\}_{\alpha \in I}$ be a family of connected subspaces of X. If for all $\alpha, \beta \in I$ we have $C_\alpha \cap C_\beta \neq \emptyset$, then $\bigcup_{\alpha \in I} C_\alpha$ is connected
- Connected component $C(x)$ of a point $x \in X$
- The connected component $C(x)$ of x is connected and closed. If for $x, y \in X$ we have $C(x) \cap C(y) \neq \emptyset$, then $C(x) = C(y)$
- Definitions of neighbourhood and locally path-connected space.
- Let X be locally path connected, then for all $x \in X$, $P(x) = C(x)$
- Corollary: Let X be locally path-connected. Then X is connected if and only if X is path-connected.

Lecture 7, 11.03.2019

- Totally disconnected space, isolated point, and first examples.
- Definition of Cantor Set.
- Proposition: The Cantor set C is totally disconnected, closed, and has no isolated points.
- Theorem: Let $X \subseteq \mathbb{R}^n$ be totally disconnected, closed, non empty, bounded, with no isolated points. Then X is homeomorphic to the Cantor set. (Not proved in the lecture)
- Theorem: Let $X \subseteq \mathbb{R}^n$ be closed and bounded, then there exists $f : C \longrightarrow X$ continuous and surjective. (Not proved in the lecture)

Lecture 8, 13.03.2019

- Definitions of compactness, open cover, subcover.
- $\mathbb{R}$ is not compact.
- Let X be a compact non empty space, and let $f : X \longrightarrow \mathbb{R}$ be a continuous function. Then f has a maximum.
- The interval $[a, b]$ is compact in $\mathbb{R}$.
- Let X be compact and $Y \subseteq X$ closed in X. Then Y is compact.
- Let X be a topological space, with basis $\mathcal{B}$. The space X is compact if and only if for every open cover $\{U_i\}_{i \in I}$ of X such that $U_i \in \mathcal{B}$ for all $i \in I$, there exists a finite subcover.
- Let $f : X \longrightarrow Y$ be continuous and surjective map. If X is compact, then Y is compact.

- Let X and Y be compact spaces, then $X \times Y$ is compact.
- A subset $X \subseteq \mathbb{R}^n$ is compact if and only if it is closed and bounded.

Lecture 9, 18.03.2019

- Infinite product, disadvantage of box topology on it.
- Product topology.
- Lemma: Let $\{X_i\}_{i \in I}$ a family of topological spaces, and let $X := \prod_{i \in I} X_i$, endowed with the product topology. Then we have:
  - the projection $\pi_j : X \longrightarrow X_j$ is continuous;
  - a map $f : Z \longrightarrow X : z \mapsto \big(f_i(z)\big)_{i \in I}$ is continuous if and only if each f_i is continuous.
- Tychonoff's theorem.

Lecture 10, 20.03.2019

- Definitions of limit of a sequence, first countable space, neighbourhood basis, Hausdorff space, converging sequence.
- Zariski topology as a non-example.
- Let X be a Hausdorff topological space, then any sequence has at most one limit.
- Let X be a metric space with metric topology, then X is first countable and Hausdorff.
- Let X be a first countable metric space, and let $A \subseteq X$. The following hold:
  - $\overline{A} = \Big\{x : \text{ exists } \{x_n\}_{n \in \mathbb{N}} \subseteq A \text{ with } x = \lim \{x_n\}\Big\}$
  - A is closed if and only if the limit of a sequence contained in A is still an element in A (if such limit exists).
- Let X, Y be topological spaces, and assume that X is first countable. Then $f : X \longrightarrow Y$ is continuous if and only if for every sequence $\{x_n\}_n \subseteq X$ with limit x, the sequence $\{f(x_n)\}_n \subseteq Y$ has limit $f(x)$.
- Definitions of sequentially compact space, totally bounded space.
- Let X be a first countable compact topological space, then X is sequentially compact.

Lecture 11, 25.03.2019

- Definitions of first and second countable space.
- Let X be second countable, if X is sequentially compact, then it is compact.
- Let X be a metric space, then the following are equivalent:
  - X is compact;
  - X is sequentially compact;
  - X is complete and totally bounded.

Lecture 13, 27.03.2019

- Definition of the space $C(X,Y)$, definition of equicontinuous family of functions.
- Ascoli-Arzelá Theorem: Let X and Y be compact metric space. Then a set $\mathfrak{F} \subseteq C(X,Y)$ is compact if and only if it is closed and equicontinuous.
- Definition of dense subset and Baire space.
- Existence of continuous nowhere differentiable functions $f : [0,1] \longrightarrow \mathbb{R}$.

Lecture 14, 01.04.2019

- Lemma: Let X be a Hausdorff space, and $C \subseteq X$ compact. Then C is closed.
- Lemma: Let X be a Hausdorff space, and $C \subseteq X$ compact. Then for every $x \in X \setminus C$ there exist open sets U, V such that $C \subseteq U$, $x \in V$ and $U \cap V = \emptyset$.
- Corollary: Let X be a compact Hausdorff space. Then $C \subseteq X$ is closed if and only if C is compact.
- One-point compactification, example: stereographic projection.
- Definitions of regular space, normal space.
- Proposition: Let X be a compact Hausdorff space, then X is normal.
- Theorem: Let X be normal and 2nd countable, then X is metrisable.

Lecture 15, 03.04.2019

- Lemma: Let X be a compact space, Y Hausdorff and $f : X \longrightarrow Y$ continuous. Then f is a closed map (i.e. for all closed subsets $C \subseteq X$, it follows that $f(C) \subseteq Y$ is closed.
- Corollary: Let X be a compact space, Y Hausdorff and $f : X \longrightarrow Y$ continuous and bijective. Then f is a homeomorphism.
- Informal introduction on Quotient Spaces.
- Proposition: Let X be a compact space, Y Hausdorff, $f : X \longrightarrow Y$ continuous and surjective. Then $U \subseteq Y$ is open if and only if $f^{-1}(U) \subseteq X$ is open.
- Proposition: Let X be a topological space, Y a set and $f : X \longrightarrow Y$ a function. Then the collection of subsets

  $$\tau := \{U \subseteq Y : f^{-1}(U) \text{ is open in } X\}$$

  defines a topology on Y.
- Remark: The topology defined above is the finest topology on Y that makes the map f continuous.
- Remark: In the same setting as above, given a point $y \in Y \setminus f(X)$, it is an isolated point.
- Gluing construction, definitions of quotient topology and quotient space.
- Proposition: Let $f : X \longrightarrow Y$ be a quotient map. Let $\sim$ be the equivalence relation on X such that, for $x, y \in X$

  $$x \sim y \Longleftrightarrow f(x) = f(y).$$

  Then the map

  $$\begin{aligned} \overline{f} : X/\sim \longrightarrow Y : \\ : [x] \mapsto f(x) \end{aligned}$$

  is well defined and is an homeomorphism between the spaces $X/\sim$ and Y.

Lecture 16, 08.04.2019

- Examples of quotient spaces, line with two zeros, sphere, cylinder, torus, Möbius strip.
- Universal property of quotient space.

Lecture 17, 10.04.2019

- Some ways of writing the sphere as a quotient space.
- Introduction of Klein Bottle, and projective plane.
- Proposition: Let X be a Hausdorff space, $\sim$ an equivalence relation on X and let the canonical projection $p : X \longrightarrow X/\sim$ be open. Then we $X/\sim$ is Hausdorff if and only if the set

$$R = \{(x, y) \in X \times X : x \sim y\}$$

is closed.
- Definition of Manifold.

Lecture 18, 15.04.2019

- Definition of loop, definition of homotopy of paths.
- Example: linear homotopy.
- Proposition: Being homotopic is an equivalence relation on paths in a given space.

Lecture 19, 17.04.2019

- Definitions of homotopy class and null homotopic loop.
- Fundamental Group.
- Proposition: The fundamental group $\pi_1(X, x_0)$ is a well defined group. The neutral element is the class of the constant loop in x_0, and the inverse of $[\alpha]$ is given by $[\alpha^{-1}]$.
- Theorem: Let X be a topological space, and $x_0, x_1 \in X$ such that there exists a path $\gamma : [0, 1] \longrightarrow X$ with $\gamma(0) = x_0$, and $\gamma(1) = x_1$. Then there exists an isomorphism

$$\begin{aligned} \beta_\gamma : \pi_1(X, x_1) &\longrightarrow \pi_1(X, x_0) \\ : [\alpha] &\mapsto [\gamma * \alpha * \gamma^{-1}] \end{aligned}$$

- Corollary: Let X be a path-connected space, then for all $x_0, x_1 \in X$ we have $\pi_1(X, x_0) \cong \pi_1(X, x_1)$.
- Basic examples of contractible spaces.

- Lemma: Let X be a topological space, and let $x_0 \in X$. Suppose that $X = \cup_{i \in I} U_i$, where U_i is a path-connected open set containing x_0 for all $i \in I$. Then every loop in X based at x_0 is homotopic to a concatenation of loops in x_0 each contained in some U_i.

- The fundamental group of the sphere S^n, for $n \geq 2$, is trivial.

Lecture 20, 29.04.2019

In this and next lecture, the fact that $\pi_1(S^1) = \mathbb{Z}$ was given as an assumption.

- Brower's fixed point theorem

- Lemma: Let $f : X \longrightarrow Y$ be a continuous map between topological spaces, such that $f(x_0) = y_0$. Then there is an induced map

$$f_* : \pi_1(X, x_0) \longrightarrow \pi_1(Y, y_o) : [\gamma] \mapsto [f \circ \gamma]$$

- Corollary: Let X and Y be homeomorphic path-connected topological spaces. Then $\pi_1(X) \cong \pi_1(Y)$.

- Corollary: The two dimensional sphere S^2 and the 2-torus T^2 are not homeomorphic.

Lecture 21, 06.05.2019

- More examples of spaces that can be distinguished using the fundamental group π_1: cylinder and Moebius strip not homeomorphic to the circle, but they are homotopically equivalent.

- Definition of homotopy between spaces. Basic example with alphabetic letters.

- Lemma:Let $f, g : X \longrightarrow Y$ be homotopic maps, and let $x_0 \in X$. Then there exists a path γ from $f(x_0)$ to $g(x_0)$ such that

$$g_* = \beta_\gamma \circ f_*.$$

- Corollary: Let $f : X \longrightarrow Y$ be a homotopy equivalence. Then the induced map f_* is an isomorphism.

- Corollary: Let X and Y be two homotopically equivalent and path connected spaces, then $\pi_1(X) \cong \pi_1(Y)$.

- Theorem: $\mathbb{R}^2$ is not homeomorphic to $\mathbb{R}^n$, for all $n \geq 3$.

Lecture 22, 08.05.2019

- Definitions of covering, lift of a map;
- Basic examples;
- Existence and uniqueness of lifts of paths and homotopies;
- Theorem: The fundamental group of the circle S^1 is $\mathbb{Z}$.

Lecture 23, 13.05.2019

- Theorem: The fundamental group of the circle S^1 is $\mathbb{Z}$ (completion of the proof);
- Theorem: Let $p : \tilde{X} \longrightarrow X$ be a covering, with $\tilde{X}$ path-connected. Let $p(\tilde{x_0}) = x_0$. Then the map

  $$\sigma : \pi_1(X, x_0) \longrightarrow p^{-1}(x_0) : [\gamma] \mapsto \tilde{\gamma}(1)$$

  which sends the homotopy class of γ to the end point of the unique lift of γ starting at $\tilde{x_0}$, is well defined and surjective. Moreover, if $\pi_1(\tilde{X}, \tilde{x_0}) = \{e\}$, then σ is injective as well.

Lecture 24, 15.05.2019

- Definition of Free Group and Free Product;
- Proposition: The free group is indeed a group;
- Proposition: Let $\{G_\alpha\}_\alpha$ be a family of groups, and $\phi_\alpha : G_\alpha \longrightarrow H$ a group homomorphism for all α. Then there exists a unique

  $$\phi : *_\alpha G_\alpha \longrightarrow H$$

  such that $\phi|_{G_\alpha} = \phi_\alpha$ for all α.
- Lemma: Let G be a group, and $A \subseteq G$. Then there exists a unique maximal (respect to inclusion) normal subgroup containing A, denoted with $<< A >>$. Moreover, the following holds:

  $$<< A >>= \{ \prod_{i=1}^{n} g_i a_i g_i^{-1} : g_i \in G, a_i \in A \}$$

  .
- Presentation of a group.

Lecture 25, 20.05.2019

- Statement of Van Kampen theorem;
- Corollary: Under the same assumptions of Van Kampen theorem, if $\pi_1(A, x_0) = \pi_1(B, x_0) = \{e\}$, then $\pi(X, x_0) = \{e\}$
- Corollary: Under the same assumptions of Van Kampen theorem, if $\pi_1(A \cap B, x_0) = \{e\}$, then

$$\pi_1(X, x_0) = \pi_1(A, x_0) * \pi_1(B, x_0)$$

- Example: Rose (or bouquet of circles);
- Fundamental group of the torus.

Lecture 26, 22.05.2019

- Continuation of fundamental group of the torus, using Van Kampen;
- Fundamental group of the Klein Bottle;
- Idea of proof of Van Kampen in the general case.