Topology — Exercise Sheet 1

Prof. Dr. Alessandro Sisto — Due to 27 February

Ex.1

Show that every open set in $\mathbb{R}$ is the union of a collection of *disjoint* open intervals (a, b), where we allow $a = -\infty$ and $b = +\infty$.

Ex.2

For each $x \in \mathbb{R}$, let $I_x = (x, \infty)$, and let $I_\infty = \emptyset$ and $I_{-\infty} = \mathbb{R}$. Check that

$$\mathcal{T} = \{I_x \mid x \in \mathbb{R} \cup \{-\infty, \infty\}\}$$

defines a topology on $\mathbb{R}$.

Ex.3

Let X be a set and let p be an element of X. Check that

$$\mathcal{T} = \{A \subseteq X \mid p \notin A \text{ or } X - A \text{ is finite}\}$$

defines a topology on X.

Ex.4

Let $X = \{a, b, c, d\}$. Which of the following are topologies for X?

(i) $\{\emptyset, X, \{a\}, \{b\}, \{a, c\}, \{a, b, c\}, \{a, b\}\}$;

(ii) $\{\emptyset, X, \{a\}, \{b\}, \{a, b\}, \{b, d\}\}$;

(iii) $\{\emptyset, X, \{a, c, d\}, \{b, c, d\}\}$

Ex.5

Let $\mathcal{T}$ be the topology for $\mathbb{R}$ described in Question 2. Which of the following functions $f \colon \mathbb{R} \to \mathbb{R}$ are continuous *with respect to* $\mathcal{T}$?

1. $f(x) = x^2$;
2. $f(x) = x^3$;
3. $f(x) = \begin{cases} 5 \text{ if } x > 5 \\ 0 \text{ otherwise;} \end{cases}$
4. $f(x) = -x$

Ex.6

In this exercise, we want to understand a little bit better continuous maps in the topology of Question 2. For this exercise, we say that a map $f : \mathbb{R} \to \mathbb{R}$ is *standard-continuous*, if it is continuous with respect to the usual topology on $\mathbb{R}$. We say that it is $\mathcal{T}$*-continuous* if it is continuous with respect to the topology described in Question 2. Let f be a function that is standard-continuous. Can you find a property that f needs to satisfy to be also $\mathcal{T}$-continuous?