Topology **Exercise Sheet 12**

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Ex.1:

Let X, Y be topological spaces, let $x_0 \in X$ and $y_0 \in Y$ be points and let $f\colon X \to Y$ be a continuous map such that $f(x_0) = y_0$. Let $f_*\colon \pi_1(X, x_0) \to \pi_1(Y, y_0)$ be the map defined as $f_*([\alpha]) = [f \circ \alpha]$. Show that f_* is well-defined and it is a group homomorphism.

Ex.2:

Prove the following statements.

(a) Let X be a path connected topological space. Show X is contractible if and only if for any path-connected topological space Y and any pair of functions $f, g\colon X \to Y$, we have that f and g are homotopic.

(b) Show that a path-connected topological space X is contractible if and only if for any path-connected topological space Y and any pair of continuous function $f, g\colon Y \to X$, we have that f and g are homotopic.

Ex.3:

Let X be the union of all straight lines of the form $ax = by$, with $a, b \in \mathbb{Z}$, equipped with the subspace topology. Show that X is path connected but not locally path connected. We recall that a topological space Z is locally path connected if every point has a basis of path connected (with respect to the induced topology) neighbourhoods.

Ex.4:

Note! This exercise is hard, and giving a complete rigorous solutions may be very long and technical. Such a solution is not required: it is ok to be a bit sloppy and focus on ideas.

Recall that a space X is contractible if there exists a contraction $C\colon X \times [0,1] \to X$ and a point x_0 such that $C(x, 0) = x$ and $C(x, 1) = x_0$ for every $x \in X$. We don't require that $C(x_0, t) = x_0$ for every possible t (this would be, in some sense, the analogous of homotopy of paths). If we add that hypothesis, that is we ask that $C(x_0, t) = x_0$ for all t, then we say that X *deformation retracts* to a point (more precisely to the point x_0). The goal of the exercise is to show that deformation retracting to a point is a stronger property than being contractible. The following exercise consists of exercises 5, 6 a) and 6 b), page 18 of the book Algebraic Topology of Allen Hatcher. In the book there are some pictures and we will refer to them.

(a) Show that if a space X deformation retracts to a point $x_0 \in X$, then for each neighborhood U of x_0 there exists a neighborhood $V \subseteq U$ of x_0 such that the inclusion map $i\colon V \to U$ is homotopic in U to the constant map c_{x_0}.

(b) Let X be the subspace of $\mathbb{R}^2$ consisting of the horizontal segment $[0,1] \times \{0\}$ together with all the vertical segments $\{r\} \times [0, 1-r]$, for r a rational number in $[0,1]$. Show that the space X deformation retracts to any $x_0 \in [0,1] \times \{0\}$, but not to any other point [See point (a)].

(c) Let Y be the subspace of $\mathbb{R}^2$ that is the union of an infinite number of copies of X arranged as in the picture in the book. Show that Y is contractible, but it does not deformation retract onto any point.

    Hint: The first step is the following. Let Z be the subspace of Y given by the thick line (look at the picture in the book, and exercise 6 (c)). Show that there is a continuous map $C\colon Y \times [0,1] \to Y$ such that $C(y,0) = y$ and $C(y,1) \in Z$ for all $y \in Y$.