Topology **Exercise Sheet 13**

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Sometimes you will probably need to say that two spaces are homotopic equivalent, but even if this is obvious from the geometric intuition point of view, writing down the explicit homotopy could be very long. You are allowed here to just say that two spaces are homotopic equivalent, and maybe add a good picture :)

Also, we recall that a space X is *simply connected* if it is path connected and $\pi_1(X) = \{1\}$.

Ex.1:

Let X be the subspace of $\mathbb{R}^3$ obtained as the union of the unit sphere and the three coordinate planes, i.e.

$$X = \{\| (x, y, z) \| = 1\} \cup \{(x, y, 0)\} \cup \{(x, 0, z)\} \cup \{0, y, z)\}.$$

Compute $\pi_1(X)$.

Ex.2:

Using Van Kampen's Theorem, compute the fundamental group of the torus.

Ex.3:

We want to give a topological proof to the Fundamental Theorem of Algebra in a simpler case: for every polynomial $p = z^n + a_1 z^{n-1} + \cdots + a_n$ with $n > 0$ and $\| a_1 \| + \cdots + \| a_n \| < 1$, show that p admits at least one root.

1. Let $f : S^1 \to \mathbb{C} - \{0\}$ be the map defined as $z \mapsto z^n$. Show that f is not null-homotopic.
2. Show that there is an homotopy H between f and the restriction of p on S^{-1}, such that H has values in $\mathbb{C} - \{0\}$. (*Hint: here we use the assumption on the coefficients.*)
3. Assume, by contradiction, that p does not admit any root. Show that this implies that p is null-homotopic in $\mathbb{C} - \{0\}$. This provides a contradiction.

Ex.4:

Let p_1, p_2 be distinct points in $\mathbb{R}^n$. Show that there is a linear map $L\colon \mathbb{R}^n \to \mathbb{R}$ such that $L(p_1) \neq L(p_2)$. Let $\{p_1, \ldots, p_n\}$ be a finite subset of $\mathbb{R}^n$, for $n \geq 3$, and let $X = \mathbb{R}^n - \{p_1, \ldots, p_n\}$. Show that $\pi_1(X) = \{1\}$.