Topology **Exercise Sheet 2**
Prof. Dr. Alessandro Sisto Due to March 6
Luca De Rosa

This exercise sheet is longer than the first one. We don't expect you to solve it from the top to the bottom (of course if you do so, it would be a good training!). So, if you have time constraints, instead of trying to do everything in a superficial way, choose a certain number of exercises, and complete them as good as you can. Fill in all the details, motivate every claim and don't be sketchy.

Ex.1:

The goal of this exercise is to give some equivalent characterisations for the interior of a set. Let X be a topological space and let Y be a subset of X. Let:

(i) $\text{int}(Y) = \{x \in X \mid \text{ there exists } O \text{ open such that } x \in O \subseteq Y\}$ (definition in the notes);

(ii) Y_1 be the maximal open that is contained in Y;

(iii) Y_2 be the union of all the open sets that are contained in Y.

Show that $\text{int}(Y) = Y_1 = Y_2$.

Ex.2:

The goal of this exercise is to give some equivalent characterizations for the closure of a set. Let X be a topological space and let Y be a subset of X. Let:

(i) $\overline{Y} = \text{int}(Y) \cup \{x \in X \mid \text{ for each open } O \text{ that contains } x, \quad O \cap Y \neq \emptyset \neq O \cap (X - Y)\}$;

(ii) Y_1 be the minimal closed set that contains Y;

(iii) Y_2 be the intersection of all the closed sets that contain Y;

(iv) $Y_3 = X - \text{int}(X - Y)$.

Show that $\overline{Y} = Y_1 = Y_2 = Y_3$.

Ex.3:

Give an example of two subsets A and B of $\mathbb{R}$ such that:

$$A \cap B = \emptyset, \quad \overline{A} \cap B \neq \emptyset, \quad A \cap \overline{B} \neq \emptyset$$

Bonus: can you find two (essentially different) such examples?

Ex.4:

Let A and B be subsets of a topological space X. Show that:

(a) $\text{int}(A) \cap \text{int}(B) = \text{int}(A \cap B)$.

(b) $\text{int}(A) \cup \text{int}(B) \subseteq \text{int}(A \cup B)$.

(c) $\overline{A} \cup \overline{B} = \overline{A \cup B}$.

(d) $\overline{A \cap B} \subseteq \overline{A} \cap \overline{B}$.

(e) Give one example where the equality in part (b) is satisfied, one where it fails, one where the equality in part (d) is satisfied and one where it fails.

Ex.5:

[There exist infinitely many primes]: Let $\mathbb{Z}$ be the set of integer numbers. For every pair of integers $a, b \in \mathbb{Z}$, with $b > 0$, let $B_{a,b}$ be the set

$$B_{a,b} = \{a + kb \mid k \in \mathbb{Z}\}.$$

Prove the following facts:

(a) The set $\mathcal{B} = \{B_{a,b} \mid a, b \in \mathbb{Z}, b > 0\}$ forms a basis for a topology $\mathcal{T}$ on $\mathbb{Z}$.

(b) For every a, b, with $b > 0$, the set $B_{a,b}$ is both open and closed in $\mathbb{Z}$ with respect to $\mathcal{B}$.

(c) Let $P = \{2, 3, \dots\}$ be the set of primes. Use the above facts to show that P needs to be infinite. *Hint! Consider the set* $\mathbb{Z} - \bigcup\{B_{0,p} \mid p \in P\}$.

Ex.6:

Let X, Y be topological spaces, and let $f\colon X \to Z$ and $g\colon Y \to W$ be maps. Then we can define a map $(f \times g\colon X \times Y \to Z \times W$ as $(f \times g)(x, y) = (f(x), g(y))$. We showed (i.e. you can find in the lecture notes) that $f \times g$ is continuous if and only if f and g are continuous. The goal of this exercise is to show that certain other properties are preserved/not preserved under products. We say that a function $f\colon X \to Z$ is *open* if for every open set $O \subseteq X$ we have that $f(O)$ is open. Similarly f is *closed* if the image of each closed set is closed.

(a) Show that if f and g are open, then so is $f \times g$;

(b) Show with a counterexample that the product of closed functions is not necessarily closed.

Ex.7:

Let (X, d) be a metric space equipped with a finite number of points. Show that in X the distance topology coincides with the discrete topology.

Ex.8:

Let p be a prime number, and $d\colon \mathbb{Z} \times \mathbb{Z} \to [0, \infty)$ be a function defined by

$$d_p(x, y) = p^{-\max\{m \in N\colon p^m | x-y\}},$$

where $p^m | x - y$ means p^m divides $x - y$. Prove that d_p is a metric on $\mathbb{Z}$ and that $d_p(x, y) \leq \max\{d_p(x, z), d_p(z, y)\}$ for every $x, y, z \in \mathbb{Z}$.

Ex.9:

Let X be a topological space equipped with a topology $\mathcal{T}_X$. Let be Y a subset of X, and let $\mathcal{T}_Y$ be the subset topology on Y with respect to $\mathcal{T}_X$. Let Z be a subset of Y, let $\mathcal{T}_{Z,Y}$ be the subset topology on Z with respect to $\mathcal{T}_Y$ and let $\mathcal{T}_{Z,X}$ be the subset topology on Z with respect to $\mathcal{T}_X$. Show that $\mathcal{T}_{Z,Y} = \mathcal{T}_{Z,X}$.

Ex.10:

Let Y be a subspace of a topological space X (i.e. Y is a topological space equipped with the subspace topology) and let A be a subset of Y. Let $\text{int}_X(A)$ be the interior of A with respect to X and $\text{int}_Y(A)$ be the interior of A with respect to Y. Show that $\text{int}_X(A) \subseteq \text{int}_Y(A)$ and give an example of when the equality does not hold.