Topology **Exercise Sheet 3**

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Questions 1 and 6 are more conceptual and should have priority. Questions 4 and 5 admit a relatively short solution. Question 7 is harder, and you should leave it as the last one.

Ex.1:

Let X and Y be topological spaces, and let A be a subset of X and B be a subset of Y. Show that

$$\overline{A} \times \overline{B} = \overline{A \times B}.$$

In particular, conclude that the product of two closed sets is a closed set.

Ex.2:

Show that the following are homeomorphic:

(i) The interval $[0, 1]$ and the interval $[2, 5]$;

(ii) The interval $(-1, 1)$ and the real line $\mathbb{R}$;

(iii) The closed disk or radius one in $\mathbb{R}^2$ and the closed square $[-1, 1] \times [-1, 1]$ is $\mathbb{R}^2$;

Ex.3:

Let $p = (-1, 0)$ and $q = (2, 0)$ be points in $\mathbb{R}^2$, and let $D_1 = \{z \in \mathbb{R}^2 \mid d(z, p) < 1\}$ and $D_2 = \{z \in \mathbb{R}^2 \mid d(z, q) < 2\}$. Which of the following are connected?

(i) $D_1 \cup D_2$;

(ii) $\overline{D_1} \cup D_2$;

(iii) $\overline{D_1} \cup \overline{D_2}$;

You may use the fact that $\overline{D_1} = \{z \in \mathbb{R}^2 \mid d(z, p) \leq 1\}$ and $\overline{D_2} = \{z \in \mathbb{R}^2 \mid d(z, q) \leq 2\}$.

Ex.4:

Let X be a set equipped with the discrete topology. Which subsets of X are connected?

Ex.5:

Let X and Y be path connected spaces. Show that $X \times Y$ is path connected.

Ex.6:

From the fact that an interval $[a, b]$ is connected, deduce the intermediate value theorem. That is, prove that for a continuous function $f \colon [a, b] \to \mathbb{R}$ and for each c such that $f(a) < c < f(b)$, there exists $x \in [a, b]$ such that $f(x) = c$.

Ex.7:

Let $S^1 = \{z \in \mathbb{R}^2 \mid d(z, (0, 0)) = 1\}$ be the unit circle in $\mathbb{R}^2$, and let $f \colon S^1 \to \mathbb{R}$ be a continuous function. Show that there exists $z \in S^1$ such that $f(z) = f(-z)$. In particular, f is not injective.

Note: if $z = (x_0, y_0)$ is a point in S^1, then $-z$ is the point $(-x_0, -y_0)$.