Topology **Exercise Sheet 4**

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Ex.1:

Let X be a topological space and A a subset of X. Show that if $A \subseteq B \subseteq \overline{A}$ and A is connected, then so is B.

Ex.2:

Show that a subset of $\mathbb{R}$ is connected if and only if it is an interval.

Ex.3:

Show that a subset of $\mathbb{R}$ is totally disconnected if and only if it does not contain any non-empty open interval.

Ex.4:

We say that a family $\mathcal{A}$ of subsets of a topological space X has the *finite intersection property* if for each (non-empty) finite subfamily $\mathcal{F}$ of $\mathcal{A}$ we have that

$$\bigcap_{A \in \mathcal{F}} A \neq \emptyset.$$

Show that a topological space X is compact if and only if for every family of closed subsets $\mathcal{A}$ that has the finite intersection property, we have that

$$\bigcap_{A \in \mathcal{A}} A \neq \emptyset.$$

Ex.5:

Let X be a compact topological space, O an open subset of X an $\{C_i\}$ be a (possibly infinite) family of closed sets such that

$$\bigcap C_i \subseteq O.$$

Show that it is possible to find a *finite* set of indices $I = \{i_1, \ldots, i_n\}$ such that

$$\bigcap_{i_j \in I} C_{i_j} \subseteq O.$$

Ex.6:

For a space X, let X' be the subspace of X obtained by removing all the isolated points of X, i.e. all the points of X which are open and closed in X. Let B_n be the subspace of $[0,1]$ that consists of all the numbers having a base 2 decimal expansion $.a_1a_2\ldots$ in which at most n of the digits a_i are 1, and let $B = \cup B_n$. Draw a picture of B_1 and B_2 and determine B'_n and B'. Deduce that there for each n there is a space X such that the sequence

$$X \supset X' \supset X'' \supset \cdots$$

becomes the empty set only after n stages.