Topology **Exercise Sheet 5**

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The Exercise 7 is the hardest, and should be left as last one. All the exercises admit rather short solution.

Ex.1:

Let $\mathcal{T}$ be the following topology on the real line $\mathbb{R}$:

- $\emptyset \in \mathcal{T}$;
- for each finite set $F \subset \mathbb{R}$, we declare $\mathbb{R} - F \in \mathcal{T}$.

(a) Check that $\mathcal{T}$ is a topology and that $(\mathbb{R}, \mathcal{T})$ is compact.

(b) Let $\mathcal{T}_{\text{std}}$ be the standard topology on $\mathbb{R}$. Show that $(\mathbb{R}, \mathcal{T})$ and $(\mathbb{R}, \mathcal{T}_{\text{std}})$ are not homeomorphic.

Ex.2:

Show that $[0,1) \times [0,1)$ is homeomorphic to $[0,1] \times [0,1)$, but not to $[0,1] \times [0,1]$.

Ex.3:

Let X_i, for $i \in I$, be a family of Hausdorff topological spaces. Show that $X = \prod_{i \in I} X_i$ is a Hausdorff space.

Ex.4:

Write down an example of a topological space that is not Hausdorff. *Note: recall that metric spaces are Hausdorff. If you have an example, check that is not a subspace of metric space (i.e. with respect to the induced topology).*

Ex.5:

Let X be a first-countable topological space, x be a point of X and $\{O_\alpha\}_{\alpha=1}^{\infty}$ be a neighborhood basis for x.

(i) For each $n \in N$, let $U_n = \bigcap_{\alpha=1}^{n} O_\alpha$, and let x_n be any point in U_n. Show that $\{x_n\}$ converges to x.

(ii) Let $\{y_i\}$ be a sequence such that for each $n \in \mathbb{N}$ and $\alpha \in \mathbb{N}$ there is an $i > n$ such that $y_i \in O_\alpha$. Show that there exists a subsequence $\{y_{i_j}\}$ of $\{y_i\}$ that converges to x.

Ex.6:

Let $(\mathbb{R}, \mathcal{T})$ be the real line equipped with the topology described in Question 1. Show that $(\mathbb{R}, \mathcal{T})$ is not first countable.

Ex.7:

Let $I = [0, 1]$ and consider the space I^I (that is, I-many copies of I). Show that I^I is compact but not sequentially compact.

Hints:

(i) You are allowed to use Tychonoff's theorem in the case of an uncountable product.

(ii) You may want to think of the space I^I as the space of functions $f\colon I \to I$.

(iii) To check that a subsequence does not converge, it is enough to show that it does not converge on a coordinate.