Topology **Exercise Sheet 6**

Prof. Dr. Alessandro Sisto
Luca De Rosa

Due to 3 April

Exercise 2 is the most important. It consists of four parts, which may be considered as four independent exercises.

Ex.1:

(a) Let X be a metric space, $x \in X$ be a point of X and $\{O_\alpha\}$ be an open cover. Show that for every O_α that contains x there exists ε such that $B_\varepsilon(x) \subseteq O_\alpha$.

(b) Let X be a compact metric space and let $\{O_\alpha\}$ be an open cover for X. Show that there exists a Lebesgue number for the cover $\{O_\alpha\}$ using the fact that X is sequentially compact.

Ex.2:

Let (X, d_X) be a compact metric space and (Y, d_Y) be a complete metric spaces, and let $C(X,Y) = \{f \colon X \to Y \mid f \text{ is continuous}\}$. We recall that the distance in $C(X,Y)$ is defined as $d(f,g) = \sup_{x \in X}\{d_Y(f(x), g(x))\}$.

(a) Show that d is indeed a metric.

(b) Show that $C(X,Y)$ is a complete metric space.

(c) Let $\mathcal{F} \subseteq C(X,Y)$ be a compact set. Show that $\mathcal{F}$ is closed and equicontinuous.

(d) Let $\mathcal{F}$ be a subset of $C(X,Y)$. Show that $\mathcal{F}$ is equicontinuous if and only if $\overline{\mathcal{F}}$ is.

Ex.3:

Let Z be a complete metric space and let Y be a subset of Z. Show that Y is complete if and only if it is closed.

Ex.4:

Let X be a compact topological space, and consider the space $C(X, \mathbb{R})$. Show that $C(X, \mathbb{R})$ is a ring.