Topology **Exercise Sheet 8**

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With this exercise sheet we want you to get more used to quotient topology. Ex.1 provides a useful source of examples and counterexamples for exercise 2, so they go together. Ex.3 is relatively easy and short. Ex.4 relates connected components and quotient topology. Finally, consider Ex.5 as an extra optional technical training.

Ex.1:

Consider the following equivalence relation on $\mathbb{R}$:

$$x \sim y \Leftrightarrow \begin{cases} x = y & \text{or} \\ |x| = |y| & \text{and } |x| > 1. \end{cases}$$

Let $X = \mathbb{R}/\sim$ equipped with the induce topology. Show that X is not an Hausdorff space and that every point of X has a neighborhood homeomorphic to the interval $(-1, 1)$.

Ex.2:

Let X, Y be topological spaces, and let $f \colon X \to Y$ be a continuous surjection. Assume that Y is equipped with the quotient topology, which means that a set $U \subseteq Y$ is open if an only if $f^{-1}(U)$ is open in X. Decide if the following are true or false: in case they are true, prove them, in case they are false, find a counterexample.

(a) If X is compact, so it is Y;

(b) If X is Hausdorff, so it is Y;

(c) If X is normal, then Y is Hausdorff;

(d) If $|X| = \infty$, then $|Y| = \infty$;

(e) If X is connected, so it is Y;

(f) If X is a metric space, so it is Y.

Ex.3:

Prove or disprove with a counterexample the following statement: *Let X be a compact space, and let $q\colon X \to Y$ be a quotient of X. Then the map q is open.*

Hint: You may want to consider the interval $[-2, 2]$ with all the points in $[-1, 1]$ identified.

Ex.4:

Let X be a topological space, and assume that all connected components of X are open. (Note, this is not always true: example $\mathbb{Q} \subseteq \mathbb{R}$ with the induced topology.) Let $q\colon X \to Y$ be a quotient of X. Show that the connected components of Y are also open.

Ex.5:

Let $q\colon X \to Y$ be a quotient and let $A \subset X$ be such that $q^{-1}(q(A)) = A$. A set A with this property is called *saturated*. Show that if q is an open map, then also $\overline{A}$ and $\mathrm{Int}(A)$ are saturated. Give an example where the map q is not open and the above is false (i.e. there is a saturated set B such that $\overline{B}$ or $\mathrm{Int}(B)$ are not saturated).