Topology | Exercise Sheet 1

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Ex.1

Show that every open set in $\mathbb{R}$ is the union of a collection of *disjoint* open intervals (a, b), where we allow $a = -\infty$ and $b = +\infty$.

Solution:

We know that every open set in $\mathbb{R}$ is the union of open intervals as above. We need to show that we can carefully choose those interval to be disjoint. Let O be an open set. If O is empty, then the claim trivially holds, because the empty set is an empty collection of intervals. So assume that O is non-empty. For each point $x \in O$, we are going to define an open interval $\mathrm{Int}(x)$ that contains x, with the property that for each $y \in \mathrm{Int}(x)$, we have that $\mathrm{Int}(y) = \mathrm{Int}(x)$.

If we can do this, then we are done. Indeed, let x, y be two points of O. Then the intervals $\mathrm{Int}(x)$ and $\mathrm{Int}(y)$ either coincide or are disjoint. Thus, the set

$$\{\mathrm{Int}(x) \mid x \in O\}$$

contains disjoints interval whose union is O.

We define $\mathrm{Int}(x)$ as the union of all the open intervals contained in O that contain x. Note that such an union is an open set. There are several ways to see this. The easiest one is to remember that in a topology the union of open sets is an open set.

We want to show that $\mathrm{Int}(x)$ is an interval, namely, if $y, z \in \mathrm{Int}(x)$, then $[y, z] \subseteq \mathrm{Int}(x)$. But this is true because, by construction, y and z are contained in an open interval that contains x (and is contained in O). Let (a, b), respectively (c, d) be such intervals. Thus we have $m = \min\{a, c\} < x < \max\{c, d\} = M$, in particular (m, M) is contained in $\mathrm{Int}(x)$ and contains $[y, z]$.

Let y be a point in $\mathrm{Int}(x)$. We need to show that $\mathrm{Int}(y) = \mathrm{Int}(x)$. Since $\mathrm{Int}(x)$ is an open interval that contains y, then $\mathrm{Int}(x) \subseteq \mathrm{Int}(y)$. But this means that $x \in \mathrm{Int}(y)$, thus $\mathrm{Int}(y) \subseteq \mathrm{Int}(x)$.

Note: One crucial property that was used in this exercise is the fact that the union of two open intervals whose intersection is non empty, is again an open interval!

Ex.2

For each $x \in \mathbb{R}$, let $I_x = (x, \infty)$, and let $I_\infty = \emptyset$ and $I_{-\infty} = \mathbb{R}$. Check that

$$\mathcal{T} = \{I_x \mid x \in \mathbb{R} \cup \{-\infty, \infty\}\}$$

defines a topology on $\mathbb{R}$.

Solution:
This is true because given I_x and I_y (assume that $x \leq y$), then $I_x \cup I_y = I_x$ and $I_x \cap I_y = I_y$. Moreover, let $\{x_i\}$ be a (possibly infinite family of elements of $\mathbb{R}$, and let $\widehat{x} = \inf\{x_i\}$. Then we have

$$\bigcup I_{x_i} = I_{\widehat{x}}.$$

Indeed, for each $y > \widehat{x}$, there must be x_j such that $y \in I_{x_j}$, and hence $I_{\widehat{x}} \subseteq \bigcup I_{x_i}$.

Conversely, let $y \in I_{\widehat{x}}$. Then $y > \widehat{x}$. Hence there is x_j such that $\widehat{x} < x_j < y$, and thus $y \in I_{x_j}$.

Ex.3

Let X be a set and let p be an element of X. Check that

$$\mathcal{T} = \{A \subseteq X \mid p \notin A \text{ or } X - A \text{ is finite}\}$$

defines a topology on X.

Solution:
We start by noticing that $X - X$ is finite and that $p \notin \emptyset$. Thus $\{X, \emptyset\} \subseteq \mathcal{T}$. We need to check that $\mathcal{T}$ is closed under finite intersection and (possibly infinite) union. For the intersection, let A_1 and A_2 be elements of $\mathcal{T}$. If at least one of them does not contain p, then the intersection $A_1 \cap A_2$ also does not contain p. So it is an element of $\mathcal{T}$. Thus, let's assume that they both contain p. This means that the sets $X - A_1$ and $X - A_2$ are both finite. Note that

$$X - (A_1 \cap A_2) = (X - A_1) \cup (X - A_2).$$

Since $X - A_1$ and $X - A_2$ are finite, so it the union $X - (A_1 \cup A_2)$. Hence $A_1 \cap A_2$ is contained in $\mathcal{T}$.

For the union, let $\{A_i\}$ be a (possibly infinite) family of elements of $\mathcal{T}$, and let $A = \bigcup A_i$ be their union. If for all i the element p is not contained in A_i, then $p \notin A$, and thus $A \in \mathcal{T}$. Conversely, suppose that there is j such that $p \in A_j$. Then $p \in A$, so we want to show that $X - A$ is finite. Note that, since $p \in A_i$, we have that $X - A_j$ is finite. Since

$$X - (\bigcup A_i) = \bigcap (X - A_i),$$

and we know that at least one element on the right hand side is finite (the one corresponding to j), we get the claim.

Ex.4

Let $X = \{a, b, c, d\}$. Which of the following are topologies for X?

(i) $\{\emptyset, X, \{a\}, \{b\}, \{a, c\}, \{a, b, c\}, \{a, b\}\}$; **yes**

(ii) $\{\emptyset, X, \{a\}, \{b\}, \{a, b\}, \{b, d\}\}$; **no**

(iii) $\{\emptyset, X, \{a, c, d\}, \{b, c, d\}\}$ **no**

Ex.5

Let $\mathcal{T}$ be the topology for $\mathbb{R}$ described in Question 2. Which of the following functions $f\colon \mathbb{R} \to \mathbb{R}$ are continuous *with respect to* $\mathcal{T}$?

1. $f(x) = x^2$; **no**
2. $f(x) = x^3$; **yes**
3. $f(x) = \begin{cases} 5 \text{ if } x > 5 \\ 0 \text{ otherwise;} \end{cases}$ **yes**
4. $f(x) = -x$ **no**

Ex.6

In this exercise, we want to understand a little bit better continuous maps in the topology of Question 2. For this exercise, we say that a map $f\colon \mathbb{R} \to \mathbb{R}$ is *standard-continuous*, if it is continuous with respect to the usual topology on $\mathbb{R}$. We say that it is $\mathcal{T}$-*continuous* if it is continuous with respect to the topology described in Question 2. Let f be a function that is standard-continuous. Can you find a property that f needs to satisfy to be also $\mathcal{T}$-continuous?

Solution:
A standard-continuous function that is monotonic increasing is also $\mathcal{T}$-continuous. Indeed, let O be an open in the $\mathcal{T}$-topology, that is $O = (a, \infty)$. We start eliminating some trivial cases.

It is clear that if $O = \emptyset$ or $f(\mathbb{R}) \cap O$ is empty then the statement is trivially true. So suppose this is not the case, that is, that $f(\mathbb{R}) \cap O \neq \emptyset$.

Since O is standard-open, then $f^{-1}(O)$ is standard-open. Moreover, since f is continuous and monotone, the preimage of an interval has to be an interval. Thus, $f^{-1}(O)$ has the form (a, b) for $a, b \in \mathbb{R} \cup \{\infty, -\infty\}$. By the cases above, $a \neq \infty$ and $a < b$. The only thing that we need to show is that $b = \infty$. Suppose that this was not true. Then there is $c > b$ such that $f(c) \notin O$ and $x \in (a, b)$ such that $f(x) \in O$. Since f is monotone, $f(x) \leq f(c)$. Moreover, $f(x) \in O$ and $f(c) \notin O$. But since O is an open interval of the form (o, ∞) for some o, we need to have $o < f(x) \leq f(c)$. Thus $f(c)$ has to be in O.