Topology **Exercise Sheet 10**

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We recall that a topological space X is *contractible* if there exists a continuous map $F\colon X \times [0,1] \to X$ and a point $p \in X$ such that for each $x \in X$ we have that $F(x,0) = x$ and $F(x,1) = p$.

Ex.1:

Let X be a convex subset of $\mathbb{R}^n$, for some $n < \infty$. Show that X is contractible.

Solution:
Choose $x \in X$. The idea is the following: define F to be the map that "slides" each point $y \in X$ towards x, in a linear way. Explicitly, we define F as the map.

$$F(y,t) = (1-t)y + t(x)$$

It is clear that F is continuous, $F(y,0) = y$ and $F(y,1) = x$ for each y. We need to argue that the image of F is contained in X. However, for each y and t, the point $F(y,t)$ is a point on the segment between x and y. Since X is convex, such a segment is contained in X.

Ex.2:

Let X be a topological space, and let γ_1, γ_2 be paths in X with the same endpoints (i.e. $\gamma_1(0) = \gamma_2(0)$ and $\gamma_1(1) = \gamma_2(1)$). Show that γ_1 and γ_2 are homotopic if and only if there is a continuous map $d\colon D^2 \to X$, such that $\gamma_1(s) = d((1-s)\pi, 1)$ and $\gamma_2(s) = d((1+s)\pi, 1)$, where the disk is parametrized in polar coordinates.

Solution:
Suppose that γ_1 and γ_2 are homotopic. This means that there exists a continuous map $F\colon [0,1] \times [0,1] \to X$ such that $F(0,t) = \gamma_1(t)$ and $F(1,t) = \gamma_2(t)$. Note that for each $s \in [0,1]$ we have that $F(s,0) = \gamma_1(0) = \gamma_2(0)$ and $F(s,1) = \gamma_1(1) = \gamma_2(1)$. We know that $[0,1] \times [0,1]$ is homeomorphic to a disk (Exercise sheet 3, Question 2.iii)), let $s\colon D^2 \to [0,1] \times [0,1]$ be such an homeomorphism. Then $d := F \circ s$ is a continuous map from the disk to X. It is not hard to see that $d((1-s)\pi, 1$ is a reparametrization of γ_1, and similarly

for γ_2. Since reparametrizing a path does not change its homotopy class, we get the result.

For the second part, let $d\colon D^2 \to X$ be as in the hypotheses. For this part of the exercise, assume that D^2 is parametrized in Euclidean coordinates. We explicitly define an homotopy F between γ_1 and γ_2 as follows:

$$F(s,t) = \begin{cases} d(\cos{(1-s)}\,\pi, (1-2t)\sin{(1-s)}\,\pi) & \text{if } t \in \left[0, \frac{1}{2}\right] \\ d(\cos{(1-s)}\,\pi, (2t-1)\sin{(1-s)}\,\pi) & \text{if } t \in \left[\frac{1}{2}, 1\right] \end{cases}$$

The idea is that at time 0, the map $F(s)$ parametrized the above arc of the disk. Then, when t increase, the map $F_t(s)$ "slides down" till it reach the lower arc of the disk.

Ex.3:

Let X be a path connected topological space. Show that $\pi_1(X) = \{1\}$ if and only if for every pair of points x, y of X, there exists only one homotopy class of paths joining them.

Solution:
Assume that $\pi_1(X) = \{1\}$ and suppose that there are two points $x, y \in X$ such that there are non-homotopic paths γ_1 and γ_2 joining them. Consider the juxtaposition $\gamma = \gamma_1 * \gamma_2^{-1}$, where with γ_2^{-1} we denote that path γ_2 with reversed orientation. Since $\pi_1(X, x) \cong \pi_1(X) = \{1\}$, and since γ is a loop based in x, we have that γ is homotopic to the trivial path on x (i.e. the path defined $c_x(t) = x$ for all t). Thus the image of γ (union the image of c_x) bounds a disk. But since $\gamma = \gamma_1 * \gamma_2^{-1}$, and since reversing the orientation does not change the image, we have that γ_1 and γ_2 bound a disk, which is a contradiction.

On the other hand, suppose that for every pair of points $x, y \in X$, we have that there is only one homotopy class of paths joining them. Let $x \in X$ be any point point. Then there exists only one homotopy class of paths between x and x. Thus, all loops based at x are homotopic. Hence $\pi_1(X, x) = \{1\}$.

Ex.4:

Let X, Y be topological spaces and let $x \in X$ and $y \in Y$ be points.

- Show that there is a map

$$p \colon \pi_1 \left(X \times Y, (x, y) \right) \to \pi_1(X, x) \times \pi_1(Y, y).$$

Solution:
Let q_X and q_Y be the projections to the first, respectively second, coordinates of $X \times Y$. It is clear that, if γ is a loop in $X \times Y$ based in (x, y), then $q_X \circ \gamma$ is a loop in X based in x and $q_Y \circ \gamma$ is a loop in Y based in y. Given an equivalence class $[\gamma] \in \pi_1(X \times Y, (x, y))$, we would like to defined $p([\gamma])$ as $([q_X \circ \gamma], [q_Y \circ \gamma])$. However, we need to show that such a map is well defined, that is we need to show that if $\gamma \sim_{X \times Y} \gamma'$, then $q_X \circ \gamma \sim_X q_X \circ \gamma'$, and similarly for Y, where $\sim_Z$ denotes being homotopic in the space Z.

We are going to show this for the space X. The space Y is completely analogous. Let $F \colon [0, 1] \times [0, 1] \to X \times Y$ be an homotopy between paths γ and γ'. Then $q_X \circ F$ is such that $q_X \circ F(s, 0) = q_X \circ \gamma(s)$ and $q_X \circ F(s, 1) = q_X \circ \gamma'(s)$. Moreover, $q_X \circ F$ is constant on $\{0\} \times [0, 1]$ and $\{1\} \times [0, 1]$. Thus we get the claim.

- Show that p is a homomorphism of groups.

Solution:
Let $[\beta]$ and $[\gamma]$ be elements of $\pi_1(X \times Y, (x, y))$. We want to show that

$$p([\beta * \gamma]) = p([\beta]) * p([\gamma]).$$

However, choosing representatives $\beta \in [\beta]$ and $\gamma \in [\gamma]$ note that (just looking at the definition of $*$ and q_X) the following holds:

$$q_X \circ (\beta * \gamma)(s) = (q_X \circ \beta) * (q_X \circ \gamma)(s).$$

The same holds for Y. Thus we get the equality by the first part.

- Show that the map p is surjective.

Solution:
Let α be a loop in X based at x, and let β be a loop in Y based at y. Then $\gamma\colon [0,1] \to X \times Y$ defined as $\gamma(s) = (\alpha(s), \beta(s))$ is a loop of $X \times Y$ based at (x, y). Note that we are using the fact that product of continuous maps is continuous. It is clear by construction that $p_X \circ \gamma = \alpha$, and similarly for β. Thus we have shown that given any pair $([\alpha], [\beta]) \in \pi_1(X, x) \times \pi_1(Y, y)$, we can find an element $[\gamma]$ such that $p([\gamma]) = ([\alpha], [\beta])$.

- Show that the map p is injective.

Solution:
We will show that the kernel of p is trivial. Let $[\gamma]$ be such that $p([\gamma]$ is trivial, let $\gamma \in [\gamma]$ be a representative and let F_X and F_Y be homotopies showing that $q_X \circ \gamma \sim_X c_x$ and $q_Y \circ \gamma \sim_Y c_y$.

Then the map $F\colon [0,1] \times [0,1] \to X \times X$ defined as

$$F(s,t) = (F_X(s,t), F_Y(s,t))$$

is such that

- $F(s,0) = (F_X(s,0), F_Y(s,0) = (q_X \circ \gamma(s), q_Y \circ \gamma(s)) = \gamma(s)$;
- $F(s,1) = (F_X(s,0), F_Y(s,0) = (c_x(s), c_y(s)) = c_{(x,y)}$;
- $F(0,t) = (F_X(0,t), F_Y(0,t)) = (x,y)$ and $F(1,t) = (F_X(1,t), F_Y(1,t)) = (x,y)$.

In particular, is an homotopy between γ and $c_{(x,y)}$. Thus the map p is injective.