Topology **Exercise Sheet 11**

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Ex.1:

Recall that X is a contractible space when the identity map on X is null-homotopic, i.e. is homotopic to some constant map in a point. Show that if X is contractible, then X is path connected and $\pi_1(X) \cong \{1\}$.

Solution:

Let $C: X \times [0,1] \to X$ be a contraction, and let $x_0 = C(x,1)$ (recall that a contraction is a map C such that $C(x,0) = x, C(x,1) = x_0$ for all x). Given two points $y, z \in X$, we define a path γ between them as follows:

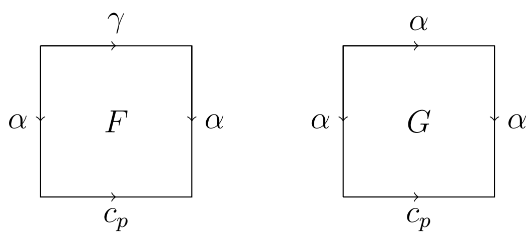
$$\gamma(s) = \begin{cases} C(y, 2s) & \text{if } s \in \left[0, \frac{1}{2}\right] \\ C(z, 2-2s) & \text{if } x \in \left[\frac{1}{2}, 1\right] \end{cases}$$

Thus, if we can show that for some $p \in X$ we have that $\pi_1(X,p) \cong \{1\}$, then the result will hold for all $x \in X$. Let $p = x_0$, and let γ be a loop based in p. Note that the map:

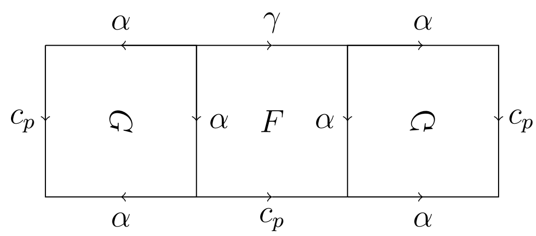
$$F(s,t) = C(\gamma(s), t)$$

is such that $F(s,0) = \gamma(s)$, $F(s,1) = p$, but we don't have (in general) that $F(0,t) = p$. Indeed, $F(0,t) = C(p,t)$ is a continuous path $\alpha: [0,1] \to X$ with $\alpha(0) = \gamma(0) = p$ and $\alpha(1) = p$. Thus, is an element of $\pi_1(X,p)$.

In a pictorial way, $F(s,t)$ if a square where on the top edge there is the path γ, on the bottom edge there is the constant path on p, and on the side edges there is the path α. If we now consider the map $G(s,t) = C(\alpha(s), t))$ we obtain a square where on all the edges a part from the bottom one, there is the path α, and on the bottom one there is the constant path on p.



Now, we can glue at each side of the F square, a copy of the G square, such that the outer lateral edge is the one labeled by c_p. This should be clear in the next picture. Note that the direction of the arrows is important (a part of the ones on the constant path c_p, since it is constant).



This should give an intuitive idea of why we can find an homotopy between $[\alpha^{-1} * \gamma * \alpha]$ and the constant path (it is clear that the class of the path $[\alpha^{-1} * c_p * \alpha]$ is the class of the constant path). We can produce explicitly an homotopy as follows:

$$H(s,t) = \begin{cases} C(\alpha(t), 1-3s) & \text{if } s \in \left[0, \frac{1}{3}\right]; \\ C(\gamma(3s-1), t) & \text{if } s \in \left[\frac{1}{3}, \frac{2}{3}\right]; \\ C(\alpha(t), 3s-2) & \text{if } s \in \left[\frac{2}{3}, 1\right]. \end{cases}$$

You can explicitly compute that H is continuous, that $H(s,0) = \alpha^{-1} * \gamma * \alpha$, that $H(s,1) = \alpha^{-1} * c_p * \alpha$, that $H(0,t) = p$ for all t and that $H(1,t) = p$ for all t. This is an homotopy that shows that for every $[\gamma] \in \pi_1(X,p)$, the conjugation $[\alpha^{-1} * \gamma * \alpha] = [c_p]$.

Note that we showed that $[\alpha^{-1}] \cdot \pi_1(X,p) \cdot [\alpha] = \{1\}$. However, since conjugation is an isomorhpism of groups, we obtain that $\pi_1(X,p) = \{1\}$. More precisely, suppose that there was $[\gamma] \neq [c_p]$. Then $[\alpha^{-1}] \cdot [\gamma] \cdot [\alpha] = [c_p] = [\alpha^{-1}] \cdot [\alpha]$. Multiplying by $[\alpha]$ on the left and by $[\alpha^{-1}]$ on the right we get a contradiction.

Ex.2:

Let $I = [0,1]$, and let X be a metric space. Let γ_1, γ_2 be paths in X with the same endpoints. Let $\mathcal{S}$ be the subset of $C(I,X)$ defined as $\{f \in C(I,X) \mid (f(0) = \gamma_1(0), f(1) = \gamma_1(1)\}$. Show that there is a path $\Gamma\colon [0,1] \to C(I,X)$ between γ_1 and γ_2 contained in $\mathcal{S}$, if and only they are homotopic (recall that $C(X,Y) = \{f\colon X \to Y \mid f \text{ is continuous}\}$ with distance defined as $d(f,g) = \sup_{x \in X}\{d_Y(f(x), g(x))\}$).

Hint: you can use Heine-Cantor Theorem.

Solution:
Let $\Gamma\colon I \to C(I,X)$ be a path between γ_1 and γ_2 that is contained in $\mathcal{S}$. We define an homotopy F between them as $F(s,t) = \Gamma(t)(s)$. Since Γ is a path contained in $\mathcal{S}$, we only need to show that $\mathcal{F}$ is continuous. Pick any $\varepsilon > 0$ and any point $(s,t) \in [0,1] \times [0,1]$. We want to show that there exists $\delta > 0$ such that if $d((s,t),(s',t')) < \delta$, then $d(F(s,t),F(s',t')) < \varepsilon$. By triangular inequality,

$$d(F(s,t),F(s',t')) \leq d(F(s,t),F(s,t')) + d(F(s,t'),F(s',t')) \leq$$
$$\leq d(\Gamma(s)(t),\Gamma(s)(t')) + d(\Gamma(s)(t'),\Gamma(s')(t')$$

Since $\Gamma(s) \in C(I,X)$ (i.e. it is a continuous function from I to X), there exists δ_1 such that if $|t-t'| < \delta_1$, then $d(\Gamma(s)(t),\Gamma(s)(t')) \leq \frac{\varepsilon}{2}$. Moreover, since Γ is continuous, we have that there exists $\delta_2 > 0$ such that if $|s-s'| < \delta_2$, then $d_{C(I,X)}(\Gamma(s),\Gamma(s')) \leq \frac{\varepsilon}{2}$. Note $d_{C(I,X)}(\Gamma(s),\Gamma(s')) < \frac{\varepsilon}{2}$ implies that $d_X(\Gamma(s)(t),\Gamma(s')(t)) \leq \frac{\varepsilon}{2}$ for all $t \in I$.

Let $\delta = \min\{\delta_1,\delta_2\}$. Then if $d((s,t),(s',t')) \leq \delta$, we have that $|s-s'| < \delta_2$ and $|t-t'| < \delta_1$. By the triangular inequality above, we get the result.

Now, suppose that γ_1 and γ_2 are homotopic via the map F, and let $F_t\colon I \to X$ be the map defined as $F_t(s) = F(s,t)$. Consider the map $\Gamma\colon I \to C(I,X)$ defined as $\Gamma(t) = F_t$. It is clear that $\Gamma(0) = \gamma_1$ and $\Gamma(1) = \gamma_2$. We claim that Γ is continuous. For every $\varepsilon > 0$ and $t \in [0,1]$, we need to find a $\delta > 0$ such that if $|t-t'| < \delta$, then $d_{C(I,X)}(\Gamma(t),\Gamma(t')) \leq \varepsilon$. Using Heine-Cantor, we have that F is uniformly continuous. In particular, there exists δ_1 such that for every $(s,t),(s't') \in [0,1] \times [0,1]$ if $d((s',t'),(s,t)) < \delta_1$, then $d(F(s,t),F(s',t')) < \varepsilon$. Fix $\delta = \frac{1}{2}\delta_1$ and let t' be such that $|t-t'| \leq \delta$. Then

$$d_{C(I,X)}(\Gamma(t),\Gamma(t')) = \sup_{s\in I}\{d(\Gamma(t)(s),\Gamma(t')(s))\} = \sup_{s\in I}\{(d(F(s,t),F(s,t'))\}.$$

By equicontinuity of F, and since $d((s,t),(s,t')) < \delta_1$, we get the result. The fact that Γ is a path contained in $\mathcal{S}$ is clear because F is an homotopy.

Ex.3:

Let $p\colon \mathbb{R} \to S^1$ be the map defined as

$$p(t) = (\cos(2\pi t), \sin(2\pi t)).$$

Show that p is a covering map.

Solution:
Let (x, y) be a point of S^1 and let $t \in \mathbb{R}$ be such that $x = \cos(2\pi t)$ and $y = \sin(2\pi t)$. It is a well known fact that $p^{-1}((x, y)) = \{t + n \mid n \in \mathbb{Z}\}$.

Let $\varepsilon = \frac{1}{10}$. We claim that $U = p((t - \varepsilon, t + \varepsilon)$ is an evenly covered open set of (x, y). Since (x, y) was arbitrary, this concludes the proof.

It is clear that $p^{-1}(U) = \bigcup_{n \in \mathbb{Z}} (t+n-\varepsilon, t+n+\varepsilon)$. By the choice of ε, all the intervals $V_n = (t+n-\varepsilon, t+n+\varepsilon)$ are disjoint. Fix $n \in \mathbb{Z}$, we want to show that $p_{|V_n}$ is an homeomorphism between V_n and U. It is clear that it is a continuous bijection. Note that V_n is contained in at least one of the following open sets: $A_m = [m, m+\frac{1}{2}], B_m = [m+\frac{1}{4}, m+\frac{3}{4}], C_m = [m+\frac{1}{2}, m+1], D_m = [m+\frac{3}{4}, m+\frac{5}{4}]$, for some $m \in \mathbb{Z}$. We define the inverse function as:

$$p_{|V_n}^{-1}(x, y) = \begin{cases} m + \frac{1}{2\pi} \arccos(x) & \text{if } V_n \subseteq A_m \\ m + \frac{1}{2\pi} \arcsin(y) & \text{if } V_n \subseteq B_m \\ m + \frac{1}{2\pi}(2\pi - \arccos(x)) & \text{if } V_n \subseteq C_m \\ m + \frac{1}{2\pi}(2\pi + \arcsin(y)) & \text{if } V_n \subseteq D_m \end{cases}$$

Since it may happen that $V_n \subseteq A_m \cap B_m$, for instance, it is not clear that the above map is well defined. However, it is a very easy computation to see that the map above is indeed the inverse of $p_{|V_n}$. Since the inverse is unique, if $V_n \subseteq A_m \cap B_m$, the definition for A_m and for B_m coincide.

Ex.4:

Let $p \colon X \to Y$ be a covering map, and let $F \colon [0, 1] \times [0, 1] \to Y$ be an homotopy between two paths. For each $y \in Y$, let U_y be an evenly covered neighbourhood of y. Show that there is $n > 0$ such that subdividing $[0, 1] \times [0, 1]$ is squares of side length $\frac{1}{n}$ we obtained that the image under F of every such sub-square is contained in U_y, for some $y \in Y$.

Solution:
Observe that $\mathcal{O} = \{F^{-1}(U_y)\}_{y \in Y}$ is an open cover of $[0, 1] \times [0, 1]$, that is, is an open cover of a compact metric space. Thus the cover $\mathcal{O}$ admits a Lebesgue number. We recall that a Lebesgue number for an open cover $\mathcal{V}$ of a metric

space X is a number $\delta > 0$ such that for each $x \in X$ we have that there exists some $V \in \mathcal{V}$ with the property that $B_\delta(x) \subseteq V$.

Let ε be a Lebesgue number for the open cover $\mathcal{O}$, and fix $n > 0$ such that $\frac{2}{n} < \varepsilon$. We claim that subdividing the square in sub-squares of side length $\frac{1}{n}$ provides the result. Let Q be such a sub-square and let x be a point of Q. We have there is some $O \in \mathcal{O}$ such that $B_\varepsilon(x) \subseteq O$. By the choice of n, we have that $Q \subseteq B_\varepsilon(x) \subseteq O$. Since $O = F^{-1}(U_y)$ for some $y \in Y$, we have that $F(Q) \subseteq U_y$), which is the desired result.

Ex.5:

Let X, Y, Z be topological spaces, and assume that $p\colon X \to Y$ and $q\colon Y \to Z$ are covering maps. Assume, moreover, that for each $z \in Z$, we have that $q^{-1}(z)$ is a finite set. Show that $q \circ p\colon X \to Z$ is a covering map.

Solution:

Let $z \in Z$, let $y_1, \ldots, y_n$ be the preimages of z under the map q (by assumption, $n < \infty$), and let U be an evenly covered neighbourhood of z. By definition of evenly covered neighbourhood, $q^{-1}(U)$ is open in Y. For each y_i, let U_i be an evenly covered open neighbourhood of y_i (with respect to the cover $p\colon X \to Y$), and let $V_i = U_i \cap q^{-1}(U)$. Note that for each V_i we have that

1. $y_i \in V_i$;
2. $q_{|V_i}$ is an homeomorphism between V_i and $q(V_i)$;
3. V_i is an evenly covered neighbourhood for $p\colon X \to Y$.

Now, let $W \subseteq Z$ be the defined as

$$W = \bigcap_{i=1}^{n} q(V_i).$$

Since n is finite, W is an open subset of Z. Moreover, since W is contained in an evenly covered neighbourhood of z (with respect to $q\colon Y \to Z$), we have that W is an evenly covered neighbourhood of z. On the other hand, since $q^{-1}(W) \subseteq \bigcup U_i$, we have that $q^{-1}(W) = \bigsqcup W_i$, where each W_i is contained in V_i, thus all the W_i are disjoint evenly-covered neighbourhoods for p. We claim that W is an evenly-covered neighbourhood of z for the cover $q \circ p$. Note that

$(q \circ p)^{-1}(V) = \bigcup p^{-1}(W_i)$. Since W_i and W_j are disjoint for $i \neq j$, we have that $(q \circ p)^{-1}(V) = \bigsqcup p^{-1}(W_i)$. Moreover, since every W_i is an evenly-covered neighbourhood for p, we have that $p^{-1}(W_i) = \bigsqcup T_i^j$, for some open sets $T_j^i \subseteq X$. Moreover, we have that $p_{|T_i^j}$ is an homeomorphism on its image, and so it is $q_{|W_i} = q_{|p(T_i^j)}$. Thus, the composition is an homeomrphims on its image, which is W. In particular, W is an evenly-covered neighbourhood for z with respect to $q \circ p$. Since z was arbitrary, this concludes the proof.