**Topology** **Exercise Sheet 12**

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# Ex.1:

Let $X, Y$ be topological spaces, let $x_0 \in X$ and $y_0 \in Y$ be points and let $f\colon X \to Y$ be a continuous map such that $f(x_0) = y_0$. Let $f_*\colon \pi_1(X, x_0) \to \pi_1(Y, y_0)$ be the map defined as $f_*([\alpha]) = [f \circ \alpha]$. Show that $f_*$ is well-defined and it is a group homomorphism.

**Solution:**

We start by showing that $f_*$ is well defined. This means that if $\alpha \sim \beta$ (where $\sim$ is the equivalence relation given by homotopy of pahts), then $f \circ \alpha \sim f \circ \beta$. Note that it is clear that $f \circ \alpha(0) = f \circ \beta(0) = f \circ \alpha(1) = f \circ \beta(1) = y_0$. Indeed, let $F\colon [0,1] \times [0,1] \to X$ be an homotopy between $\alpha$ and $\beta$. Then $f \circ F$ is a map from $[0,1] \times [0,1]$ to $Y$ such that

1. $f \circ F(s, 0) = f(\alpha(s)) = f \circ \alpha(s)$ for all $s$;
2. $f \circ F(s, 1) = f(\beta(s)) = f \circ \beta(s)$ for all $s$;
3. $f \circ F(0, t) = f(\alpha(0)) = f \circ \alpha(0) = y_0$ for all $t$;
4. $f \circ F(1, t) = f(\alpha(1)) = f \circ \alpha(1) = y_0$ for all $t$.

In particular, $f \circ F$ is an homotopy between $f \circ \alpha$ and $f \circ \beta$. Thus $f_*$ is well defined.

Now, we want to show that it is a group homomorphism. We recall that $\alpha * \beta$ is the defined as:

$$\alpha * \beta(s) = \begin{cases} \alpha(2s) & \text{if } s \in \left[0, \frac{1}{2}\right], \\ \beta(2s-1) & \text{if } s \in \left[\frac{1}{2}, 1\right]. \end{cases}$$

It is not hard to see that for every function $f$, we have that $(f \circ \alpha) * (f \circ \beta)(s) = f \circ (\alpha * \beta)(s)$ for every $s$ (just plug it in the definition). Thus we obtain:

$$\begin{aligned} f_*([\alpha]) \cdot f_*([\beta]) = [f \circ \alpha] \cdot [f \circ \beta] = [(f \circ \alpha) * (f \circ \beta)] = \\ = [f \circ (\alpha * \beta)] = f_*([\alpha * \beta]) = f_*([\alpha] \cdot [\beta]). \end{aligned}$$

In particular, $f_*$ is a group homomorphism.

# Ex.2:

Prove the following statements.

(a) Let $X$ be a path connected topological space. Show $X$ is contractible if and only if for any path-connected topological space $Y$ and any pair of functions $f, g\colon X \to Y$, we have that $f$ and $g$ are homotopic.

**Solution:**
Suppose that $X$ is contractible, and let $C\colon X \times [0,1] \to X$ be a contraction for $X$. Let $p = C(x,1)$, let $y_1 = f(p)$ and let $y_2 = g(p)$. Finally, let $\gamma$ be a path in $Y$ between $y_1$ and $y_2$. Since $Y$ is path connected, such a $\gamma$ always exists. Let $F\colon X \times [0,1] \to Y$ be the map defined as:

$$F(x,s) = \begin{cases} f(C(x,3s)) & \text{if } s \in \left[0, \frac{1}{3}\right] \\ \gamma(2s-1) & \text{it } s \in \left[\frac{1}{3}, \frac{2}{3}\right] \\ g(C(x,3-3s)) & \text{it } s \in \left[\frac{2}{3}, 1\right]. \end{cases}$$

It is clear that $F$ is continuous and such that $F(x,0) = f(x), F(x,1) = g(x)$. In particular, $F$ is an homotopy between $f$ and $g$.

Now suppose that the right hand side holds. Let $Y = X$ and let $f = \mathrm{Id}$ and $g = c_{x_0}$ for some $x_0 \in X$. Then we get that there is an homotopy $F\colon X \times [0,1] \to X$ such that $F(x,0) = x$ for every $x \in X$ and $F(x,1) = c_{x_0}(x) = x_0$. Thus $F$ is the desired contraction.

(b) Show that a path-connected topological space $X$ is contractible if and only if for any path-connected topological space $Y$ and any pair of continuous function $f, g\colon Y \to X$, we have that $f$ and $g$ are homotopic.

**Solution:**
Suppose that $X$ is contractible, and let $C\colon X \times [0,1] \to X$ be a contraction for $X$. Let $F\colon Y \times [0,1] \to X$ be the map defined as:

$$F(y,s) = \begin{cases} C(f(y),2s) & \text{if } s \in \left[0, \frac{1}{2}\right] \\ C(g(y),2s-1) & \text{it } s \in \left[\frac{1}{2}\right]. \end{cases}$$

It is clear that $F$ is continuous and such that $F(y,0) = f(y), F(y,1) = g(y)$. In particular, $F$ is an homotopy between $f$ and $g$.

Now, suppose that the right hand side holds. We want to show that $X$ is contractible. As before, let $Y = X$ and let $f = \mathrm{Id}$ and $g = c_{x_0}$ for some $x_0 \in X$. Then we get that there is an homotopy $F\colon X \times [0,1] \to X$ such that $F(x,0) = x$ for every $x \in X$ and $F(x,1) = c_{x_0}(x) = x_0$. Thus $F$ is the desired contraction.

## Ex.3:

Let $X$ be the union of all straight lines of the form $ax = by$, with $a, b \in \mathbb{Z}$, equipped with the subspace topology. Show that $X$ is path connected but not locally path connected. We recall that a topological space $Z$ is locally path connected if every point has a basis of path connected (with respect to the induced topology) neighbourhoods.

**Solution:**
We start by showing that $X$ is path connected. Indeed, every point of $X$ lies on a line through the point $(0,0)$, thus every point can be connected to the origin. Concatenating every two such paths, we get that every pair of points can be connected with a path.

Now, we want to show that $X$ is not locally path connected. Choose a point $z \in X$ and assume that $Z \neq (0,0)$. Let $\mathcal{B}$ be a neighbourhood basis for $z$. We claim that there is a neighbourhood $V \in \mathcal{B}$ that is not locally path connected. First, we can assume that $(0,0) \notin V$. Indeed, otherwise this would imply that $(0,0)$ is contained in every open set that contains $z$. Thus $X$ is not Hausdorff, but this is a contradiction since subspaces of Hausdorff are Hausdorff. This means that $V \subseteq \mathbb{R}^2 - \{(0,0)\}$. Let $\tau \in \mathbb{R}$ be any number and let $\lambda_\tau = \{(x,y) \mid y = \tau x\}$ be the line of slope $\tau$.It is not hard to see that, for any $\tau$, the line $\lambda_\tau$ separates $\mathbb{R}^2$ into two components. This means that if two points are on "different sides" of $\lambda_\tau$, then every path joining them needs to intersect $\lambda_\tau$. A precise argument for this can be obtained by applying a rotation that sends $\lambda_\tau$ to the $x$-axis, and use the intermediate value theorem. We claim that there is an irrational number $\tau$ and points $v_1, v_2 \in V$ such that $v_1$ and $v_2$ lie on different sides of $\mathbb{R}^2 - \lambda_\tau$. This would show that $X$ is not locally path connected. Indeed, since $X \subseteq \mathbb{R}^2 - (\lambda_\tau - \{(0,0)\})$, we have that every path joining such $v_1$ and $v_2$ must pass through the origin. However, since $V \subseteq (\mathbb{R}^2 - \{(0,0)\}$, this cannot happen inside $V$, which shows that $V$ is not locally path connected.

Since $V$ is a neighbourhood of $z$, there is an open set $U \subseteq V$ that contains $z$. Since the open balls form a basis of $\mathbb{R}^2$, we have that there is some $\varepsilon > 0$ such that $B_\varepsilon(z) \subseteq U \subseteq V$. Since $\mathbb{Q}$ is dense in $\mathbb{R}$, we can assume that $\varepsilon \in \mathbb{Q}$. By definition of $X$, there are $a, b \in \mathbb{Z}$ such that $z = (x, \frac{a}{b}x)$ for some $x$ (note, it may happen that $b = 0$. In that case, just apply a rotation and do the same procedure in the new coordinates). Let $z' = (x, \frac{2a+\varepsilon}{2b}x)$. We have that $z' \in V$. Since $\mathbb{R} - \mathbb{Q}$ is dense in $\mathbb{R}$, we have that there exists an irrational number $\tau$ such that $\frac{a}{b} < \tau < \frac{2a+\varepsilon}{2}$. Setting $v_1 = z$ and $v_2 = z'$ we get that $V$ is not path connected.

# Ex.4:

*Note! This exercise is hard, and giving a complete rigorous solutions may be very long and technical. Such a solution is not required: it is ok to be a bit sloppy and focus on ideas.*

Recall that a space $X$ is contractible if there exists a contraction $C\colon X \times [0,1] \to X$ and a point $x_0$ such that $C(x, 0) = x$ and $C(x, 1) = x_0$ for every $x \in X$. We don't require that $C(x_0, t) = x_0$ for every possible $t$ (this would be, in some sense, the analogous of homotopy of paths). If we add that hypothesis, that is we ask that $C(x_0, t) = x_0$ for all $t$, then we say that $X$ *deformation retracts* to a point (more precisely to the point $x_0$). The goal of the exercise is to show that deformation retracting to a point is a stronger property than being contractible. The following exercise consists of exercises 5, 6 a) and 6 b), page 18 of the book Algebraic Topology of Allen Hatcher. In the book there are some pictures and we will refer to them.

(a) Show that if a space $X$ deformation retracts to a point $x_0 \in X$, then for each neighborhood $U$ of $x_0$ there exists a neighborhood $V \subseteq U$ of $x_0$ such that the inclusion map $i\colon V \to U$ is homotopic in $U$ to the constant map $c_{x_0}$.

**Solution:**
Let $C\colon X \times [0,1] \to X$ be a contraction for $X$, and let $U \subseteq X$ be a neighborhood of $x_0$. By definition of neighborhood, $U$ contains an open set $O$ that contains $x_0$ (recall that in general neighborhoods do not need to be open). Since $C$ is continuous, we have that $C^{-1}(O) \subseteq X \times [0,1]$ is an open set that contains $(x_0, 1)$. Note that more it is true. Since $C(x_0, t) = x_0$ for every $s$, we have that $\{x_0\} \times [0,1] \subseteq C^{-1}(O)$. Recall that the product topology is defined in such that every point $(x, t)$ has an

(open) neighbourhood basis of the form $O_x \times I_t$, where $O_x$ is an open of $X$ that contains $x$ and $I_t$ is an open of $[0,1]$ that contains $t$.

Since this forms a basis and since $C^{-1}(O)$ is open, for every $(x_0, t) \in \{x_0\} \times [0,1]$ we can find $V_t \subseteq X$ and $I_t \subseteq [0,1]$ open such that $x_0 \in V_t$, $t \in I_t$ and $V_t \times I_t \subseteq C^{-1}(O)$.

It is not hard to see that $\{x_0\} \times [0,1]$ with the induced topology is homeomorphic to $[0,1]$. In particular, it is compact. Thus, we can find a finite set $\{t_1, \dots, t_n\}$ such that:

$$\{x_0\} \times [0,1] \subseteq \bigcup_{i=1}^{n} V_{t_i} \times I_{t_i} \subseteq C^{-1}(O).$$

Let $V = \bigcap_{i=1}^{n} V_{t_i}$. Since $n < \infty$, we have that $V$ is an open set of $X$ that contains $x_0$. Thus is a neighbourhood of $x_0$. By construction, $V \times [0,1] \subseteq \bigcup_{i=1}^{n} V_{t_i} \times I_{t_i} \subseteq C^{-1}(O)$. This means that $C(V, [0,1]) \subseteq O \subseteq U$. This provides the desired homotopy from the inclusion $V \to X$ to the constant map $c_{x_0}$ in the subspace $U$.

(b) Let $X$ be the subspace of $\mathbb{R}^2$ consisting of the horizontal segment $[0,1] \times \{0\}$ together with all the vertical segments $\{r\} \times [0, 1-r]$, for $r$ a rational number in $[0,1]$. Show that the space $X$ deformation retracts to any $x_0 \in [0,1] \times \{0\}$, but not to any other point [See point (a)].

**Solution:**
First, let $x_0 = (x, 0)$. Then we can explicitly define a retraction as:

$$C((u,v),t) = \begin{cases} (u, (1-2t)v) & \text{if } t \in \left[0, \frac{1}{2}\right], \\ ((2t-1)x + (2-2t)u, 0) & \text{if } t \in \left[\frac{1}{2}, 1\right]. \end{cases}$$

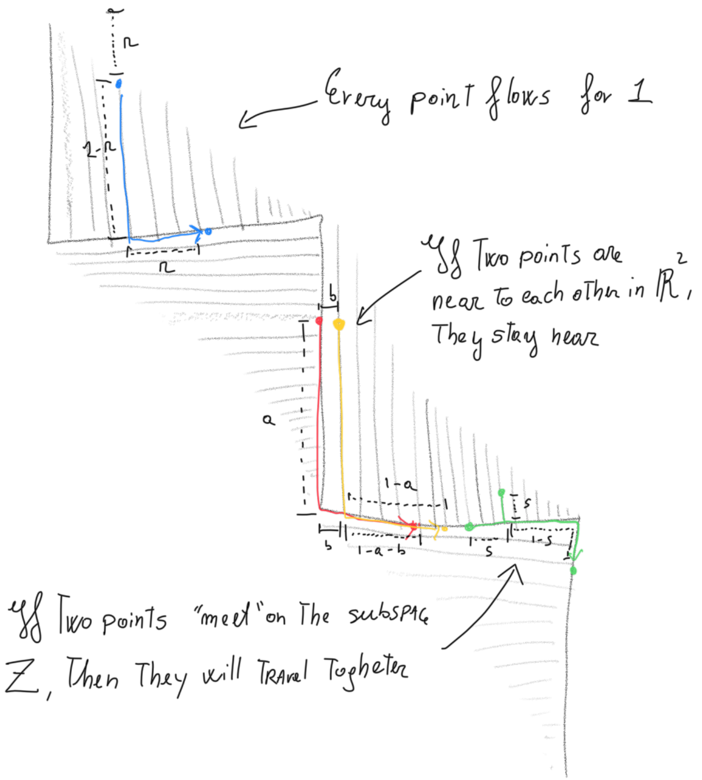
Now suppose that $x_0 = (x, y)$ for some $y \neq 0$. Let $U$ be the neighborhood of $x_0$ obtained intersecting $X$ with the ball of radius $\frac{y}{2}$ around $x_0$. It is clear that $U$ is not path connected. In particular, there is a point $z = (x_z, y_z)$, with $x_z \neq x$ such that every path between $z$ and $x_0$ exits $U$. Thus, there cannot be a neighborhood $V$ of $x_0$ contained in $U$ that can be contracted to a point inside $U$ (because otherwise the map $\gamma(t) = C(z, t)$ would be a path from $z$ to $x_0$ inside $U$). In particular, $X$ does not reformation retract onto $x_0$.

(c) Let $Y$ be the subspace of $\mathbb{R}^2$ that is the union of an infinite number of copies of $X$ arranged as in the picture in the book. Show that $Y$ is contractible, but it does not deformation retract onto any point.

*Hint: The first step is the following. Let $Z$ be the subspace of $Y$ given by the thick line (look at the picture in the book, and exercise 6 (c)). Show that there is a continuous map $C\colon Y \times [0,1] \to Y$ such that $C(y,0) = y$ and $C(y,1) \in Z$ for all $y \in Y$.*

**Solution:**
Using the same argument of part (b), we can easily get that $Y$ does not deformation retract onto any point. Indeed, every neighborhood of every point is not path connected. We want to show that $Y$ is contractible. Let $Z$ be the thick zig-zag line in the picture. We want to show that there is a continuous map $C\colon Y \times [0,1] \to Y$ such that $C(y,0) = y$ and $C(y,1) \in Z$ for all $y \in Y$. If we can do that, then we can just contract $Z$ to any point, and obtain the desired homotopy. Note that the map $C$ will not have the property that $C(z,t) = z$ for all $z \in Z$. It is worth to remark that the map that "collapses all the branches to $Z$" (i.e. apply the above point to all the copies of $X$) is not continuous. Indeed, it will fix, say, an horizontal segment of $Z$, but move all the shorter segments above it. This cannot be done in a continuous way, since those shorter segments are arbitrarily near to $Z$.

The intuitive idea of the map $C$ is that for every point $y \in Y$, we intuitively know what does it means to "flow rightwards for 1". A bit more precisely, we move with unitary speed each point on a strand in the direction of $Z$, and every point on $Z$ we move rightwards. So a general point on a strand will first go to $Z$ (in general this will take less than 1), and then flow rightwards. What you can imagine is that the space $Y$ is some kind of rope with a lot of branching, and then you pull it rightwards for the length 1. Hopefully a more precise intuition can be provided by the next picture.




It should be clear that after flowing everything by 1, the space $Y$ will be contained in $Z$. It is less clear that such a map is continuous. You can probably believe that if two points are near (in $\mathbb{R}^2$), then they are near after applying the map. More precisely, for every $s \in [0,1]$ and $y_1, y_2 \in Y$ we have that $d_{\mathbb{R}^2}(y_1, y_2) \geq d_{\mathbb{R}^2}(C(y_1, s), C(y_2, s))$. Moreover, since the map "flows" along lines, for each $y \in Y$ and $s_1, s_2 \in [0,1]$ we have that $d_{\mathbb{R}^2}(C(y, s_1), C(y, s_2)) \leq |s_1 - s_2|$. We claim that this implies continuity.

If the topology on $Y$ was given as the distance-topology with the distance

$d_{\mathbb{R}^2}$, then for every pair of points $(y_1, s_1), (y_2, s_2)$ we have:

$$d_{\mathbb{R}^2}(C(y_1, s_1), C(y_2, s_2)) \leq d_{\mathbb{R}^2}(C(y_1, s_1), C(y_2, s_1)) + d_{\mathbb{R}^2}(C(y_2, s_1), (y_2, s_2)) \leq \\ \leq d_{\mathbb{R}^2}(y_1, y_2) + |s_1 - s_2| \leq d_{\mathbb{R}^2 \times [0,1]}((y_1, s_1), (y_2, s_2)),$$

which implies continuity. However, since $Y$ is equipped with the subspace topology, if we declare a distance on $Y$ simply as $d_Y(y_1, y_2) = d_{\mathbb{R}^2}(y_1, y_2)$, we have that $\{y \in Y : d_Y(y, y_0) < r\} = B_r(y) \cap Y$. Thus, the subspace topology and the distance topology induce the same neighbourhood basis at every point. Thus they coincide. This shows that the map $C$ is continuous.