Topology **Exercise Sheet 13**

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Sometimes you will probably need to say that two spaces are homotopic equivalent, but even if this is obvious from the geometric intuition point of view, writing down the explicit homotopy could be very long. You are allowed here to just say that two spaces are homotopic equivalent, and maybe add a good picture :)

Also, we recall that a space X is *simply connected* if it is path connected and $\pi_1(X) = \{1\}$.

Ex.1:

Let X be the subspace of $\mathbb{R}^3$ obtained as the union of the unit sphere and the three coordinate planes, i.e.

$$X = \{\| (x,y,z) \| = 1\} \cup \{(x,y,0)\} \cup \{(x,0,z)\} \cup \{0,y,z)\}.$$

Compute $\pi_1(X)$.

Solution:

It is easy to see that X is path connected, and we will not do it explicitly.

Let $Y = \{(x,y,z) \in X \mid \| (x,y,z) \| \leq 1\}$ be the space obtained intersecting X with the closed unit ball. First, note that Y is homotopic equivalent to X. An homotopy is given by

$$H((x,y,z),t) = \begin{cases} (x,y,z) & \text{if } (x,y,z) \in Y \\ \frac{(x,y,z)}{(1-t)+t\|(x,y,z)\|} & \text{else} \end{cases}$$

Thus, we only need to compute the fundamental group of Y. Now, consider $A = Y \cap \{(x,y,z) \mid x < \frac{1}{3}\}$ and $B = Y \cap \{(x,y,z) \mid x > -\frac{1}{3}\}$. It is not hard to see that the intersection between A and B is homotopic equivalent to a disk. Thus, if we can show that A and B are simply connected, then so it is X. Note that A and B are (homotopic equivalent to) a disk union the upper hemisphere of a sphere union two "crossing walls" that divide the interior of such a "half sphere" in 4 parts.

Let $A_1 = A \cap \{(x,y,z) \mid y > \frac{-1}{3}\}$ and $A_2 = A \cap \{(x,y,z) \mid y < \frac{1}{3}\}$. Again, the intersection of A_1 and A_2 is homotopic equivalent to a disk (the shape is the one of half a disk), and each of those is homotopic equivalent to "half" of

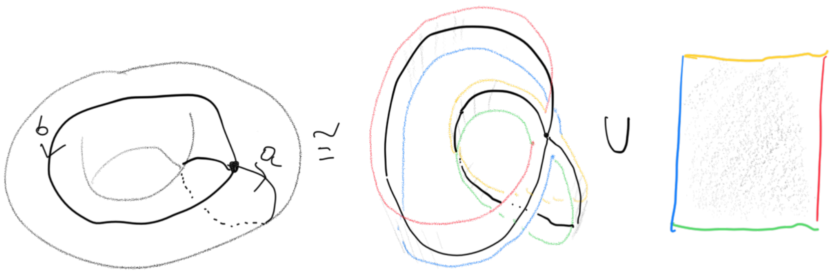
A, that is to 2 of the 4 parts in which A is divided. Subdividing again with respect to the last coordinate, we obtain two homeomorphic spaces that are homotopic equivalent to a 2-sphere. Since the 2-sphere is simply connected. By the previous observations, we have that $\pi_1(X) = \{1\}$.

A more intuitive picture is that when we consider the space Y, we can think of it as made of 8 parts, each of which is homeomorphic to a sphere (note that some small amount of details is needed to make those parts open). Since they all intersect along disks, we get the result.

Ex.2:

Using Van Kampen's Theorem, compute the fundamental group of the torus.

Solution:
We recall that the torus can be written as $A \cup B$ as in the picture.



On the right hand side, the sets A and B. The sides of B are glued corresponding to the colors (since they are open, there must be some overlap). The intersection $A \cap B$ is homotopy equivalent to the boundary of the square, that is to a circle.
It is important to fix the basepoint. Remember that $A \cap B$ looks like a "picture frame" such that every edge has a color, as the set B. We can "move" the overlap a little bit, so the the black dot on the torus lies on our favorite corner of the frame $A \cap B$. We choose it to be on the top left corner, that is the yellow-blue corner. Let x_0 be such a basepoint.

It is easy to see that B is homotopic equivalent to the rose R_2. Let i_A and i_B be the inclusions of $A \cap B$ in A and B respectively. Since $\pi_1(B, x_0) = \{1\}$, we have that i_{B*} has to be the trivial map. Thus, the formula simplifies and we have that

$$\pi_1(T, x_0) = \frac{\pi_1(A, x_0)}{\langle\langle i_{A*}(\pi_1(A \cap B, x_0))\rangle\rangle}.$$

From the observations before we know that $\pi_1(A, x_0) \cong F_2$ and $\pi_1(A \cap B, x_0) \cong \mathbb{Z}$. We need to understand the maps. Let a and b be the generators of $\pi_1(A, x_0)$ as showed in the picture. Let t be the generator of $\pi_1(A \cap B, x_0)$ obtained by going once clocwise around $A \cap B$. It is easy to see from the picture that the image of t under i_A follows the colors: yellow - red - green - blue. This corresponds to the concatenation $aba^{-1}b^{-1}$. Note that the orientation is important. Thus, the image of $\pi_1(A \cap b, x_0)$ in $\pi_1(A, x_0)$ is the group generated by the image of the generator t, that is $i_{A*}(\pi_1(A \cap B, x_0)) = \langle aba^{-1}b^{-1} \rangle$. Thus:

$$\pi_1(T, x_0) \cong \frac{F_2}{\langle\langle aba^{-1}b^{-1} \rangle\rangle} \cong \mathbb{Z}^2.$$

Ex.3:

We want to give a topological proof to the Fundamental Theorem of Algebra in a simpler case: for every polynomial $p = z^n + a_1 z^{n-1} + \cdots + a_n$ with $n > 0$ and $\| a_1 \| + \cdots + \| a_n \| < 1$, show that p admits at least one root.

1. Let $f \colon S^1 \to \mathbb{C} - \{0\}$ be the map defined as $z \mapsto z^n$. Show that f is not null-homotopic.

Solution:
It is easy to see that the map $(\mathbb{C} - \{0\}) \times [0, 1] \to S^1$ defined as $(z, t) \mapsto \frac{z}{(1-t)+t\|z\|}$ is an homotopy equivalence between $\mathbb{C} - \{0\}$ to S^1 that is constant on S^1. Thus $\pi_1(\mathbb{C} - \{0\}) = \mathbb{Z}$. Since the equivalence is constant on S^1, the class of f in $\pi_1(S^1, (1, 0))$ is (up to choose the correct basepoint) the class of the loop $t \mapsto (\cos(2\pi nt), i\sin(2\pi nt))$, which, for $n > 0$, is not a trivial element of $\pi_1(S^1, (1, 0))$. Since non-triviality of an element does not depend on the basepoint (see solution of Question 4 b), Exercise sheet 12), we have that f does not represent the trivial element in $\pi_1(S^1)$. Thus it is not null-homotopic in S^1 and hence in $\mathbb{C} - \{0\}$.

2. Show that there is an homotopy H between f and the restriction of p on S^{-1}, such that H has values in $\mathbb{C} - \{0\}$. (*Hint: here we use the assumption on the coefficients.*)

Solution:
Consider the homotopy

$$H(z,t) = z^n + t(a_1 z^{n-1} + \cdots + a_0).$$

It is clear that H is an homotopy between f and p. We need to show that for every z, t, $H(z,t) \neq 0$. We will show that $\| H(s,t) \| > 0$, which implies the claim. Recall that $z \in S^1$. We have:

$$\| H(z,t) \| = \| z^n + t(a_1 z^{n-1} + \cdots + a_n) \| \geq \| z^n \| - \| t(a_1 z^{n-1} + \cdots + a_0) \| \geq$$
$$\geq 1 - t(\| a_1 z^{n-1} \| + \cdots + \| a_n \|) \geq 1 - t(\| a_1 \| + \cdots + \| a_n \|) > 0$$

3. Assume, by contradiction, that p does not admit any root. Show that this implies that p is null-homotopic in $\mathbb{C} - \{0\}$. This provides a contradiction.

Solution:
Note that the image $p(\mathbb{C})$ is contained in $\mathbb{C} - \{0\}$. In particular, this is true for $p(B^2)$, where B^2 is the closed unit ball in $\mathbb{C}$. This means that $p_{|S^1}$ bounds a disk in $\mathbb{C} - \{0\}$ (see Exercise Sheet 10, Question 2), thus is null-homotopic.

We could also give an explicit homotopy: $H(z,t) = p((1-t)z)$.

The case without the assumption on the coefficient is just an algebraic trick to reduce to the case above. You can find it on Munkres, Theorem 56.1

Ex.4:

Let p_1, p_2 be distinct points in $\mathbb{R}^n$. Show that there is a linear map $L \colon \mathbb{R}^n \to \mathbb{R}$ such that $L(p_1) \neq L(p_2)$.

Solution:
Since $p_1 \neq p_2$, there is a coordinate in which they differ, let this coordinate be x_j. Then the map L is given by the projection on the j-th coordinate.

Let $\{p_1, \ldots, p_n\}$ be a finite subset of $\mathbb{R}^n$, for $n \geq 3$, and let $X = \mathbb{R}^n - \{p_1, \ldots, p_n\}$. Show that $\pi_1(X) = \{1\}$.

Solution:

We will proceed by induction. First suppose that $n = 1$. Then $\mathbb{R}^n - \{p_1\}$ is homotopic equivalent to the sphere $S^{n-1} \cong \{x \in \mathbb{R}^n \mid d(x, p_1) = 1\}$ (to see this, consider $H(x,t) = \frac{x-p_1}{(1-t)+t\|x-p_1\|} + p_1$). Since $\pi_1(S^{n-1}) = \{1\}$ when $n-1 \geq 2$, we get the result. Now, assume by induction hypothesis that the result holds for every subset of $\mathbb{R}^n$ consisting of at most $n-1$ points, and consider $\{p_1, \ldots p_n\}$. We can assume that $p_1 \neq p_2$ (otherwise we are done by induction), and consider a linear map L as in the part above. Up to exchange p_1 and p_2, assume that $L(p_2) > L(p_1)$.

Consider the open sets $A = L^{-1}((L(p_1), \infty)) \cap X$ and $B = L^{-1}(-\infty, L(p_1)) \cap X$. We have that $A \cap B = L^{-1}((L(p_1), L(p_2)) \cap X$. In particular, each of A, B and $A \cap B$ is an open subset of X. Since L is linear, each one of A, B and $A \cap B$ is homeomorphic to $\mathbb{R}^n - \{k \text{ points}\}$ with $k \leq n-1$. Thus by induction, each of those is simply connected. Then Van Kampen's Theorem gives that X is simply connected.