Topology **Exercise Sheet 2**

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Ex.1:

The goal of this exercise is to give some equivalent characterizations for the interior of a set. Let X be a topological space and let Y be a subset of X. Let:

(i) $\text{int}(Y) = \{x \in X \mid \text{ there exists } O \text{ open such that } x \in O \subseteq Y\}$ (definition in the notes);

(ii) Y_1 be the maximal open that is contained in Y;

(iii) Y_2 be the union of all the open sets that are contained in Y.

Show that $\text{int}(Y) = Y_1 = Y_2$.

Solution:
We show $Y_2 \subseteq Y_1 \subseteq \text{int}(Y) \subseteq Y_2$.

For the first inclusion, let $x \in Y_2$. Then x belong to an open set O that is contained in Y. We claim that $O \subseteq Y_1$. Suppose that this is not the case, then $O \cup Y_1$ would be an open set contained in Y which is strictly bigger than Y_1, which is a contradiction.

For the second inclusion, Y_1 is an open set that is contained in Y. Thus for each $x \in Y_1$, we have that $x \in \text{int}(Y)$.

For the third inclusion, let $x \in \text{int}(Y)$. Then there exists O such that $x \in O \subseteq Y$. By definition, $O \subseteq Y_2$, and thus $x \in Y_2$.

Ex.2:

The goal of this exercise is to give some equivalent characterizations for the closure of a set. Let X be a topological space and let Y be a subset of X. Let:

(i) $\overline{Y} = \text{int}(Y) \cup \{x \in X \mid \text{ for each open } O \text{ that contains } x, \quad O \cap Y \neq \emptyset \neq O \cap (X - Y)\}$;

(ii) Y_1 be the minimal closed set that contains Y;

(iii) Y_2 be the intersection of all the closed sets that contain Y;

(iv) $Y_3 = X - \text{int}(X - Y)$.

Show that $\overline{Y} = Y_1 = Y_2 = Y_3$.

Solution:

We will show that the first three items are the same as Y_3

Case $\overline{Y} = Y_3$. Let x be a point of X. Then exactly one of the following three possibilities holds:

1. There exists an open set O such that $x \in O \subseteq Y$;

2. There exists an open set O such that $x \in O \subseteq X - Y$;

3. For every open set O that contains x, O intersects both Y and $X - Y$.

Indeed, it is clear that (3) holds if and only if (1) and (2) are both false. If (1) holds, then $x \in Y$, and thus (2) cannot hold. The same apply if (2) holds. We defined $\overline{Y}$ to be the set of points that satisfy either (1) or (3). Moreover, the set of points that satisfy (2) is defined to be $\text{int}(X-Y)$. Thus $\overline{Y} = X - \text{int}(X-Y)$.

Case $Y_1 = Y_3$. If Y_1 is the minimal closed set that contains Y, then $X - Y_1$ is the maximal open set that does not contain Y. In particular, $X - Y_1$ is the maximal open set that is contained in $X - Y$. By the previous exercise $X - Y_1 = \text{int}(X - Y)$, which is equivalent to the desired equality.

Case $Y_2 = Y_3$. If Y_2 is the intersection of all the closed sets that contain Y, then $X - Y_2$ is the union of all the open set that do not contain Y. Then the conclusion follows as in the case before.

Ex.3:

Give an example of two subsets A and B of $\mathbb{R}$ such that:

$$A \cap B = \emptyset, \quad \overline{A} \cap B \neq \emptyset, \quad A \cap \overline{B} \neq \emptyset$$

Bonus: can you find two (essentially different) such examples?

Solution:

We saw in the lecture that $\partial\mathbb{Q} = \mathbb{R}$. Thus $A = \mathbb{Q}$ and $B = \mathbb{R} - \mathbb{Q}$ works. Another example is $A = [0, 1) \cup [2, 3)$ and $B = [1, 2)$.

Ex.4:

Let A and B be subsets of a topological space X. Show that:

(a) $\text{int}(A) \cap \text{int}(B) = \text{int}(A \cap B)$.

Solution:
We start with the inclusion $\text{int}(A) \cap \text{int}(B) \subseteq \text{int}(A \cap B)$. By definition, the interior of Y is the union of all the open sets that are contained in Y. Thus if a point x is contained in the left hand side, then there are open sets O_A, resp. O_B contained in A, resp. B, that contain x. Thus $O = O_A \cap O_B$ is an open set contained in $A \cap B$ that contains x. So x is a point of the interior of $A \cap B$.

For the other inclusion, let $x \in \text{int}(A \cap B)$. Then there is an open O that contains x such that $O \subseteq A \cap B$. In particular $O \subseteq A$ and $O \subseteq B$. Thus $x \in \text{int}(A) \cap \text{int}(B)$.

(b) $\text{int}(A) \cup \text{int}(B) \subseteq \text{int}(A \cup B)$.

Solution:
Let x be a point in $\text{int}(A) \cup \text{int}(B)$. Then there is an open O that contains x such that O is contained in either A or B. In particular, O is contained in $A \cup B$. Thus the conclusion follows.

(c) $\overline{A} \cup \overline{B} = \overline{A \cup B}$.

Solution:
By definition, the closure of a set Y is the intersection of all the closed sets that contain Y. Since $\overline{A \cup B}$ is closed and contains both A and B, we have that $\overline{A} \subseteq \overline{A \cup B}$ and similarly $\overline{B} \subseteq \overline{A \cup B}$. Thus we obtain the inclusion $\overline{A} \cup \overline{B} \subseteq \overline{A \cup B}$. For the other inclusion, we show that $X - \overline{A} \cup \overline{B} \subseteq X - \overline{A \cup B}$. Recall that $X - (\overline{A} \cup \overline{B}) = (X - \overline{A}) \cap (X - \overline{B})$. By the previous exercises, the inclusion that we need to show is equivalent to the following:

$$\text{int}(X - A) \cap \text{int}(X - B) \subseteq \text{int}(X - (A \cup B)) = \text{int}\big((X - A) \cap (X - B)\big).$$

The result follows from part (1).

(d) $\overline{A\cap B}\subseteq \overline{A}\cap\overline{B}$.

> **Solution:**
> Since $A\cap B\subseteq A$ and $A\cap B\subseteq B$, we have that $\overline{A\cap B}\subseteq\overline{A}$ and $\overline{A\cap B}\subseteq\overline{B}$. Thus the conclusion follows.

(e) Give one example where the equality in part (b) is satisfied, one where it fails, one where the equality in part (d) is satisfied and one where it fails.

> **Solution:**
> The equality are trivially satisfied if $A=B$.
>
> An example where equality (b) fails is the following: $A=[0,1]$ and $B=[1,2]$. Then $\text{int}(A\cup B)=(0,2)$ but $\text{int}(A)\cup\text{int}(B)=(0,1)\cup(1,2)$.
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> An example where equality (d) fails is the following: let $A=(0,1)$ and $B=(1,2)$. Then $A\cap B=\emptyset$, but $\overline{A}\cap\overline{B}=\{1\}$.

Ex.5:

[There exist infinitely many primes]: Let $\mathbb{Z}$ be the set of integer numbers. For every pair of integers $a,b\in\mathbb{Z}$, with $b>0$, let $B_{a,b}$ be the set

$$B_{a,b}=\{a+kb\mid k\in\mathbb{Z}\}.$$

Prove the following facts:

(a) The set $\mathcal{B}=\{B_{a,b}\mid a,b\in\mathbb{Z},b>0\}$ forms a basis for a topology $\mathcal{T}$ on $\mathbb{Z}$.

> **Solution:**
> We need to show that
>
> (a) For every point $n\in\mathbb{Z}$, there is $B_{a,b}\in\mathcal{B}$ such that $n\in B_{a,b}$. for this, just notice that $B_{a,1}=\mathbb{Z}$, for every $a\neq 0$.
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> (b) Given $B_{a,b}$, $B_{a',b'}$ and a point $n\in B_{a,b}\cap B_{a',b'}$, there is $B_{a'',b''}$ such that $n\in B_{a'',b''}\subseteq B_{a,b}\cap B_{a',b'}$.
>
> For the first point, just notice that $B_{a,1}=\mathbb{Z}$, for every $a\neq 0$. For the second, we claim that choosing $a''=n$ and $b''=bb'$ works. Indeed, $n\in$

$B_{n,bb'}$. We show now that $B_{n,bb'} \subseteq B_{a,b}$. The other case is completely analogous. Since $n \in B_{a,b}$, there exists $k \in \mathbb{Z}$ such that $n = a + kb$. Thus, for every $s \in \mathbb{Z}$, we have $n + sbb' = a + kb + sbb' = a + (k + sb')b \in B_{a,b}$.

(b) For every a, b, with $b > 0$, the set $B_{a,b}$ is both open and closed in $\mathbb{Z}$ with respect to $\mathcal{B}$.

Solution:
For each a, b (we always assume $b > 0$, I will stop to write it), we have that $B_{a,b}$ is open by definition. We need to show that is also closed, namely that it can be written as $\mathbb{Z} - O$, where O is an open set. If $b = 1$, then we are done, since the empty set is open. If $b > 1$, then $B_{a,b}$ consists of all numbers that can be reached from a adding a multiple of b. In particular, the first number contained in $B_{a,b}$ after a is going to be $a + b$. Thus all the numbers $a + 1, a + 2, \dots, a + (b - 1)$ are not in $B_{a,b}$. We claim that for each $0 < r < b$, the sets $B_{a,b}$ and $B_{a+r,b}$ are disjoint. Indeed, suppose there was k and k' such that $a + kb = a + r + k'b$. Then we would have $r = (k - k')b$, which is impossible by choice of r. Moreover, it is clear that $B_{a,b} \cup B_{a+1,b} \cup \cdots \cup B_{a+(b-1),b} = \mathbb{Z}$. This implies that

$$B_{a,b} = \mathbb{Z} - \bigcup_{s=1}^{b-1} B_{a+s,b}.$$

That is, we wrote $B_{a,b}$ as $\mathbb{Z}$ minus an open set. Thus $B_{a,b}$ is closed.

(c) Let $P = \{2, 3, \dots\}$ be the set of primes. Use the above facts to show that P needs to be infinite. *Hint! Consider the set* $\mathbb{Z} - \bigcup\{B_{0,p} \mid p \in P\}$.

Solution:
If the set P was finite, then $\bigcup\{B_{0,p} \mid p \in P\}$ would be the finite union of closed sets, thus a closed set. Then, $\mathbb{Z} - \{B_{0,p} \mid p \in P\}$ would be open. However, by definition of prime number we have that $\mathbb{Z} - \{B_{0,p} \mid p \in P\} = \{1, -1\}$, which is not open. Thus P has to be infinite.

Ex.6:

Let X, Y be topological spaces, and let $f\colon X \to Z$ and $g\colon Y \to W$ be maps. Then we can define a map $(f \times g\colon X \times Y \to Z \times W$ as $(f \times g)(x, y) = (f(x), g(y))$. We showed (i.e. you can find in the lecture notes) that $f \times g$ is continuous if and only if f and g are continuous. The goal of this exercise is to show that certain other properties are preserved/not preserved under products. We say that a function $f\colon X \to Z$ is *open* if for every open set $O \subseteq X$ we have that $f(O)$ is open. Similarly f is *closed* if the image of each closed set is closed.

(a) Show that if f and g are open, then so is $f \times g$;

Solution:
Let O be an open of $X \times Y$. By definition of the product topology, $O = \bigcup U_i \times V_i$ where U_i are open sets of X and V_i are open sets of Y. Then

$$f \times g(O) = f \times g\left(\bigcup U_i \times V_i\right) = \bigcup f \times g(U_i \times V_i) = \bigcup f(U_i) \times g(V_i)$$

Since the right hand side consists of a union of open sets (product of opens is open by defintion of product topology), we get that $(f \times g)(O)$ is open.

(b) Show with a counterexample that the product of closed functions is not necessarily closed.

Solution:
Let $f, g\colon \mathbb{R} \to \mathbb{R}$ be functions defined as $f(x) = x$, $g(x) = 0$. It is easy to see that f and g are closed function. However the function $f \times g\colon (x, y) \mapsto (x, 0)$ is not closed. Indeed let C be the graphic in $\mathbb{R}^2$ of the function $\frac{1}{x}$, for $x \in \mathbb{R} - \{0\}$. Then $(f \times g)(C) = (\mathbb{R}^2) \times \{0\}$, which is not a closed subset. Thus if we can prove that C is a closed subset of $\mathbb{R}^2$, then we would have that $f \times g$ is not closed.

To see that C is closed, consider the function $\eta\colon \mathbb{R}^2 \to \mathbb{R}$ defined as $\eta((x, y)) = xy$. Clearly, η is a continuous function. Note moreover that $C = \eta^{-1}(\{1\})$, in particular, it is closed since preimage of a closed set.

The fact that the preimage (under a continuous map) of a closed set is a closed set is a well know fact that you don't need to prove. However, since repetition is useful, we recall the proof of it. Let $f\colon X \to Y$ be continuous and $C \subseteq Y$ closed. Then $Y - C$ is open in Y, thus $f^{-1}(Y - C)$ is open in X. However $X - f^{-1}(Y - C) = \{x \in X \mid f(x) \notin Y - C\} = \{x \in X \mid f(x) \in C\} = f^{-1}(C)$. Thus $f^{-1}(C)$ is closed in X.

Ex.7:

Let (X, d) be a metric space equipped with a finite number of points. Show that in X the distance topology coincides with the discrete topology.

Solution:
Let D be the set of the possible distances between different points of X, that is:

$$D = \{d(x, y) \colon x, y \in X, x \neq y\}.$$

Note that X is a finite set of positive numbers. In particular, X has a minimum, which is a positive number. Let c be such a minimum. Then for every $x \in X$, the ball of radius $\frac{c}{2}$ around x contains only x. Thus, for every $x \in X$, we have that $\{x\}$ is open.

Ex.8:

Let p be a prime number, and $d \colon \mathbb{Z} \times \mathbb{Z} \to [0, \infty)$ be a function defined by

$$d_p(x, y) = p^{-\max\{m \in N \colon p^m | x - y\}},$$

where $p^m | x - y$ means p^m divides $x - y$. Prove that d_p is a metric on $\mathbb{Z}$ and that $d_p(x, y) \leq \max\{d_p(x, z), d_p(z, y)\}$ for every $x, y, z \in \mathbb{Z}$.

Solution:
The symmetry condition is trivial. By $p > 0$ and $m \in \mathbb{N}$, we have $p^{-m} > 0$ for every $m \in \mathbb{N}$ with 0 as the limit. Indeed, $d_p(x, y)$ can only be zero, if $pm | x - y$ for all natural numbers m. This can only happen if $x - y = 0$, thus $d_p(x, y) = 0$ if and only if $x = y$. For the triangle inequality condition, consider three points x, y and z. For brevity define $m_{x,z}$ as $-\log_p(d_p(x, y))$. Without loss of generality we can assume that that $m_{x,z}$ is not larger than $m_{y,z}$ (i.e. $d_p(x, z) \geq d_p(y, z)$). Then $p^{m_{x,z}}$ divides both $x - z$ and $z - y$ and therefore also $x - z + z - y = x - y$. This means that $m_{x,y} \geq m_{x,z}$

or equivalently $d_p(x,y) \leq d_p(x,z) = \max\{d_p(x,z), d_p(y,z)\}$. In particular $d_p(x,y) \leq d_p(x,z) + d_p(y,z)$ holds.

Ex.9:

Let X be a topological space equipped with a topology $\mathcal{T}_X$. Let be Y a subset of X, and let $\mathcal{T}_Y$ be the subset topology on Y with respect to $\mathcal{T}_X$. Let Z be a subset of Y, let $\mathcal{T}_{Z,Y}$ be the subset topology on Z with respect to $\mathcal{T}_Y$ and let $\mathcal{T}_{Z,X}$ be the subset topology on Z with respect to $\mathcal{T}_X$. Show that $\mathcal{T}_{Z,Y} = \mathcal{T}_{Z,X}$.

Solution:
Let $O \in \mathcal{T}_{Z,Y}$. Then $O = V \cap Z$, where $V \in \mathcal{T}_Y$. But, by definition of subset topology on Y, we have that $V = U \cap Y$, where $U \in \mathcal{T}_X$. Thus, $O = (U \cap Y) \cap Z = U \cap Z$, because $Z \subseteq Y$. Hence we get $\mathcal{T}_{Z,Y} \subseteq \mathcal{T}_{Z,X}$.

For the other inclusion, let $O \in \mathcal{T}_{Z,X}$. Then $O = U \cap Z$, where $U \in \mathcal{T}_X$. However, $V = U \cap Y$ is an open of $\mathcal{T}_Y$, and hence $V \cap Z = O$ is an open of $\mathcal{T}_{Z,Y}$.

Ex.10:

Let Y be a subspace of a topological space X (i.e. Y is a topological space equipped with the subspace topology) and let A be a subset of Y. Let $\text{int}_X(A)$ be the interior of A with respect to X and $\text{int}_Y(A)$ be the interior of A with respect to Y. Show that $\text{int}_X(A) \subseteq \text{int}_Y(A)$ and give an example of when the equality does not hold.

Solution:
Let x be a point in $\text{int}_X(A)$. Then there is an open set O of X that contains x and that is contained in A. Note that $O \cap Y$ is an open set of Y that contains x and is contained in A. Thus $\text{int}_X(A) \subseteq \text{int}_Y(A)$.

The other inclusion does not hold in general. An example is given by choosing Y to be not an open set of X, and setting $A = Y$. Indeed, A is open in Y (the interior of an open set is the open set itself), but is not open in X. Concretely, let $X = \mathbb{R}$ and $A = Y = [0,1]$. Then $[0,\varepsilon)$ is a family of opens subsets of Y that contains 0 and is contained in A. Thus 0 is in the Y-interior of A. However, 0 is not in the $\mathbb{R}$-interior of $[0,1]$.