Topology **Exercise Sheet 3**

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Questions 1 and 6 are more conceptual and should have priority. Questions 4 and 5 admit a relatively short solution. Question 7 is harder, and you should leave it as the last one.

Ex.1:

Let X and Y be topological spaces, and let A be a subset of X and B be a subset of Y. Show that

$$\overline{A} \times \overline{B} = \overline{A \times B}.$$

In particular, conclude that the product of two closed sets is a closed set.

Solution:
We start by showing $\overline{A \times B} \subseteq \overline{A} \times \overline{B}$. In order to do that, we will show that $X \times Y - \overline{A} \times \overline{B} \subseteq X \times Y - \overline{A \times B}$.

Let $(x, y) \in (X \times Y - \overline{A} \times \overline{B})$. Then we have $x \notin \overline{A}$ or $y \notin \overline{B}$ (or both). Assume that the first one holds (the other case is analogous). Then there exists an open set O of X such that $x \in O$ and $O \cap A = \emptyset$. By definition of product topology, we have that $O \times Y$ is an open in $X \times Y$. By construction, $(x, y) \in O \times Y$ and $O \times Y \cap A \times B = \emptyset$. Thus $(x, y) \in X \times Y - \overline{A \times B}$.

For the other inclusion, let $(x, y) \in \overline{A} \times \overline{B}$, and let O be an open of $X \times Y$ such that $(x, y) \in O$. We want to show that $O \cap A \times B \neq \emptyset$. By definition of product topology, we have that $O = \bigcup_i V_i \times U_i$, where the V_i are open sets of X and the U_i are open sets of Y. Since $x \in \overline{A}$, we have that for each i, $V_i \cap A \neq \emptyset$. Similarly for y. Thus, for each i we have that $V_i \times U_i \cap A \times B \neq \emptyset$. In particular, $O \cap A \times B \neq \emptyset$.

Ex.2:

Show that the following are homeomorphic:

(i) The interval $[0, 1]$ and the interval $[2, 5]$;

Solution:
Consider the map $f\colon [0,1] \to [2,5]$ defined as $f(t) = 2+3t$. It is clear that f is continuous and it's inverse is continuous. Indeed, we have $f^{-1}(t) = \frac{t-2}{3}$. Since f^{-1} is continuous, f is a homeomorphism.

(ii) The interval $(-1,1)$ and the real line $\mathbb{R}$;

Solution:
Consider the map $f\colon (-1,1) \to \mathbb{R}$ defined as

$$f(t) = \begin{cases} \frac{t}{1-t} & \text{if } t \geq 0 \\ \frac{t}{t+1} & \text{if } t \leq 0 \end{cases}$$

Since $\frac{t}{1-t} = \frac{t}{t+1}$ if $t = 0$, the map is continuous. It is not hard to see that f is bijective. The limit of f for $t \to 1$, resp. $t \to -1$, is ∞, resp. $-\infty$, and $f(0) = 0$. A simple exercise is derivatives tells us that f is strictly increasing in $f_{|(0,1)}$ and in $f_{|(-1,0)}$. Thus f is bijective. The inverse of f is

$$f^{-1}(t) = \begin{cases} \frac{t}{1+t} & \text{if } t \geq 0 \\ \frac{t}{1-t} & \end{cases}$$

which is a continuous function. Thus f is an homeomorphism.

(iii) The closed disk or radius one in $\mathbb{R}^2$ and the closed square $[-1,1] \times [-1,1]$ is $\mathbb{R}^2$;

Solution:
We want to produce an homeomorphism from the closed unit disk $B = \{x \in \mathbb{R}^2 \mid d(x,(0,0)) \leq 1\}$ to the square $Q = [-1,1] \times [-1,1]$. The idea is the following: we will "stretch" the disk to the square. To to this, it is convenient to work in polar coordinates. Let θ be the angular coordinate and ρ be the radial one. Note that for each value of θ, there only one value ρ_B such that (θ, ρ_B) is a point of the boundary of the disk (and the value is $\rho_B = 1$), and only one value ρ_Q such that (θ, ρ_Q) is a point of the boundary of the square. This amount to say that the ray starting at $(0,0)$ with angle θ intersects each of the boundaries of B and Q in exactly

one point. Let g be the function defined as $g(\theta) = \frac{\rho_Q}{\rho_B} = \rho_Q$. We claim that g is continuous. Indeed, we can explicitly write g as:

$$g(\theta) = \begin{cases} \frac{1}{\cos(\theta)} & \text{if } \theta \in [-\frac{\pi}{4}, \frac{\pi}{4}] \\ \frac{1}{\sin(\theta)} & \text{if } \theta \in [\frac{\pi}{4}, \frac{3}{4}\pi] \\ -\frac{1}{\cos(\theta)} & \text{if } \theta \in [\frac{3}{4}\pi, \frac{5}{4}\pi] \\ -\frac{1}{\sin(\theta)} & \text{if } \theta \in [\frac{5}{4}\pi, \frac{7}{4}\pi] \end{cases}$$

Let $f\colon B \to Q$ be the function defined as $f(\theta, \rho) = (\theta, g(\theta)\rho)$. The function f is a bijection from B to Q. Indeed, f stretches each ray of the disk $\{(\theta, \rho) \mid \rho \leq 1\}$ to the ray of the square $\{(\theta, \rho)\colon \rho \leq g(\theta)\}$. The inverse of f is clearly defined as $f^{-1}(\theta, \rho) = \left(\theta, \frac{\rho}{g(\theta)}\right)$ (note that g is always positive). Since g is continuous, we have that f and f^{-1} are also continuous.

Ex.3:

Let $p = (-1, 0)$ and $q = (2, 0)$ be points in $\mathbb{R}^2$, and let $D_1 = \{z \in \mathbb{R}^2 \mid d(z, p) < 1\}$ and $D_2 = \{z \in \mathbb{R}^2 \mid d(z, q) < 2\}$. Which of the following are connected?

(i) $D_1 \cup D_2$;

(ii) $\overline{D_1} \cup D_2$;

(iii) $\overline{D_1} \cup \overline{D_2}$;

You may use the fact that $\overline{D_1} = \{z \in \mathbb{R}^2 \mid d(z, p) \leq 1\}$ and $\overline{D_2} = \{z \in \mathbb{R}^2 \mid d(z, q) \leq 2\}$.

Solution:
Note that, by definition of the topology on $\mathbb{R}^2$, the sets D_1 and D_2 are open. It is easy to see that $D_1 \cap D_2 = \emptyset$. Indeed by triangular inequality, for each point $z \in \mathbb{R}^2$ we have $d(z, p) + d(z, q) \geq d(p, q) = 3$. If $d(z, p) < 1$, then $d(z, q) \leq 3 - d(z, p) > 2$, and similarly for the other case. Thus the first set is the disjoint union of open sets, hence not connected. The second and the third are connected. To see this, we will show that they are path connected. Clearly for each point $z \in D_1$, there exists a path contained in D_1 that connects z to p. This can be easily believed by staring at a picture, or writing an explicit

formula in polar coordinates. Similarly, each point $z \in D_2$ can be connected to q inside D_2.

Now we claim that there exists a path that connects p and q that is contained in both $\overline{D_1} \cup D_2$ and $\overline{D_1} \cup \overline{D_2}$. This implies that both of the unions above are path connected.

Let $\gamma\colon [0,3] \to \mathbb{R}^2$ be the path defined as $\gamma(t) = (-1+t, 0)$. Clearly γ is a continuous path that connects p and q. We claim that the image of γ is contained in $\overline{D_1} \cup D_2$ (and hence in $\overline{D_1} \cup \overline{D_2}$). For each t, if $t \leq 1$, then $d(\gamma(t), p) = d((t-1, 0), (-1, 0)) = \sqrt{(t-1+1)^2} = t$, thus $\gamma(t) \in \overline{D_1}$. Otherwise, if $t > 1$, then $d(\gamma(t), q) = d((t-1, 0), (2, 0)) = \sqrt{(t-1-2)^2} = |t-3|$. Since $1 < t \leq 3$, we have that $d(\gamma(t), q) < 2$, and thus $\gamma(t) \in D_2$.

Ex.4:

Let X be a set equipped with the discrete topology. Which subsets of X are connected?

Solution:
The only connected subsets are the points of X, i.e. the subsets $\{x\}$, for $x \in X$. Since the set $\{x\}$ cannot be written as the disjoint union of *any* pair of non-empty sets, the points of X are trivially connected.

On the other hand, let $Y \subseteq X$ be a subset that is not a point, that is, Y contains at least two elements. Let $x \in Y$. Then $\{x\}$ and $Y - \{x\}$ are disjoint open sets, because in the discrete topology every set is open. In particular, Y is not connected.

Ex.5:

Let X and Y be path connected spaces. Show that $X \times Y$ is path connected.

Solution:
Let (x_1, y_1) and (x_2, y_2) be points in $X \times Y$. We want to find a continuous map $f\colon [0,1] \to X \times Y$ such that $f(0) = (x_1, y_1)$ and $f(1) = (x_2, y_2)$.

Since X and Y are both path connected, there exist continuous functions $\gamma\colon [0,1] \to X$, $\delta\colon [0,1] \to Y$ such that $\gamma(0) = x_1$, $\gamma(1) = x_2$, $\delta(0) = y_1$

and $\delta(1) = y_2$. Let $f = \gamma \times \delta$. Since the product of continuous functions is continuous, then f is continuous. Moreover

$$f(0) = (\gamma(0), \delta(0)) = (x_1, y_1) \text{ and } f(1) = (\gamma(1), \delta(1)) = (x_2, y_2).$$

Ex.6:

From the fact that an interval $[a, b]$ is connected, deduce the intermediate value theorem. That is, prove that for a continuous function $f\colon [a, b] \to \mathbb{R}$ and for each c such that $f(a) < c < f(b)$, there exists $x \in [a, b]$ such that $f(x) = c$.

Solution:
Suppose that this was not the case. Then $[a, b] = f^{-1}((-\infty, c)) \cup ((c, \infty))$. Since $f(a) < c$ and $f(b) > c$, the two sets $f^{-1}((-\infty, c))$ and $((c, \infty))$ are non empty, and since f is continuous they are open. Moreover, the union has to be disjoint: for each point $y \in [a, b]$, either $f(y) > c$ or $f(y) < c$. Thus, we wrote $[a, b]$ as a disjoint union of non-empty, open sets, which is a contradiction.

Ex.7:

Let $S^1 = \{z \in \mathbb{R}^2 \mid d(z, (0, 0)) = 1\}$ be the unit circle in $\mathbb{R}^2$, and let $f\colon S^1 \to \mathbb{R}$ be a continuous function. Show that there exists $z \in S^1$ such that $f(z) = f(-z)$. In particular, f is not injective.

Note: if $z = (x_0, y_0)$ is a point in S^1, then $-z$ is the point $(-x_0, -y_0)$.

Solution:
To simplify notation, let's assume that S^1 is parametrized in polar coordinates by the angle $\theta \in \mathbb{R}$, with the usual convention that θ and θ' represent the same point if $\theta - \theta'$ is a multiple of 2π.

Let's consider $f(0)$ and $f(\pi)$. If they are the same, then we are done. Otherwise, consider the function $g\colon [0, \pi] \to \mathbb{R}$ defined as:

$$g(\theta) = f(\theta) - f(-\theta).$$

It is clear that $g(0)$ and $g(\pi)$ have different sign, and that the function g is continuous. Thus, by the intermediate value theorem, there is θ such that $g(\theta) = 0$, which concludes the proof.