Topology **Exercise Sheet 4**

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Ex.1:

Let X be a topological space and A a subset of X. Show that if $A \subseteq B \subseteq \overline{A}$ and A is connected, then so is B.

Solution:
We know from the lecture that if A is connected, so it is $\overline{A}$. Suppose that B is not connected. Then there exist open sets O_1 and O_2 of X such that the following holds. If we denote $B_1 = (B \cap O_1)$ and $B_2 = (B \cap O_2)$, then $B = B_1 \sqcup B_2$ and B_1 and B_2 are both non-empty. If we had that A intersects both of them, since $A \subseteq B$, then we would have $A = (A \cap O_1) \sqcup (A \cap O_2)$, which is a contradiction. So, without loss of generality, $A \subseteq B_1$. Consider the closed set $C = \overline{A} - O_2$. By construction, $A \subseteq C$, but $B_2 \cap C = \emptyset$. Thus, we found a closed set that contains A which is strictly smaller than $\overline{A}$, which is a contradiction.

Ex.2:

Show that a subset of $\mathbb{R}$ is connected if and only if it is an interval.

Solution:
Let I be an interval of $\mathbb{R}$. Since I is path connected, then it is connected.

On the other hand, let C be a connected subset of $\mathbb{R}$. If C consists of a single point, then C is an interval. In particular, C has at least two points. We recall that a set I is an interval if for any pair of points $a < b$ contained in I and for every $a < c < b$, we have $c \in I$. Suppose that C is not an interval. In particular, there exist $a < c < b$ with $a, b \in C$ and $c \notin C$. Thus, $C = \big(C \cap (-\infty, c)\big) \sqcup \big(C \cap (c, \infty)\big)$, which contradicts the fact that C is connected. Thus, C is an interval.

Ex.3:

Show that a subset of $\mathbb{R}$ is totally disconnected if and only if it does not contain any non-empty open interval.

Solution:
If a set contains a non-empty open interval I, then it is clearly not totally disconnected, since the connected component of any point $x \in I$ would contain I, but for a totally disconnected space, the connected component of a point is the singleton of the point itself. On the other hand, if a set C is not totally disconnected, then there is a connected component A of C that has more than one point. By the previous exercise, A is an interval. If A is an open interval, then we are done. Otherwise, one of the following three has to happen:

(i) $A = [a, b]$;

(ii) $A = (a, b]$;

(iii) $A = [a, b)$.

In each of the above three cases, we must have that a and b are different, otherwise A would contain at most one element. In particular, the interval (a, b) is not empty and contained in $A \subseteq C$. Thus, C contains a non-empty open interval.

Ex.4:

We say that a family $\mathcal{A}$ of subsets of a topological space X has the *finite intersection property* if for each (non-empty) finite subfamily $\mathcal{F}$ of $\mathcal{A}$ we have that

$$\bigcap_{A \in \mathcal{F}} A \neq \emptyset.$$

Show that a topological space X is compact if and only if for every family of closed subsets $\mathcal{A}$ that has the finite intersection property, we have that

$$\bigcap_{A \in \mathcal{A}} A \neq \emptyset.$$

Solution:
Assume that X is compact, and suppose that there exists a family $\mathcal{A}$ of closed subsets of X such that the intersection of all the elements of A is empty, but any finite sub collection of A has non empty intersection. For a set A, we denote by A^c the complement of A, that is: $A^c = X - A$. Since the intersection of all the elements $A \in \mathcal{A}$ is empty, we have that

$$\bigcup_{A \in \mathcal{A}} A^c = X.$$

In particular, $\mathcal{A}^c = \{A^c \mid A \in \mathcal{A}\}$ is an open cover of X. Thus, there exists a finite subfamily $\mathcal{F}^c$ of $\mathcal{A}^c$ that still covers X. We denote an element of $\mathcal{F}^c$ by A^c, where $A \in \mathcal{A}$ (that is, we underline the open set of which they are the complement). However, this implies

$$\bigcap_{A^c \in \mathcal{F}^c} A = \emptyset,$$

which is a contradiction.

On the other hand, assume that X is not compact. Then there exists an infinite family of open sets $\mathcal{A}^c$ that covers X such that for every finite subfamily $\mathcal{F}^c$ of $\mathcal{A}^c$, the union of the elements of $\mathcal{F}^c$ do not cover X. Using the strategy above we see that $\mathcal{A}$ (that is, the family of complements of $\mathcal{A}^c$) has the finite intersection property, but the intersection of all its elements is the empty set.

Ex.5:

Let X be a compact topological space, O an open subset of X an $\{C_i\}$ be a (possibly infinite) family of closed sets such that

$$\bigcap C_i \subseteq O.$$

Show that it is possible to find a *finite* set of indices $I = \{i_1, \ldots, i_n\}$ such that

$$\bigcap_{i_j \in I} C_{i_j} \subseteq O.$$

Solution:
Note that $O^c = X - O$ is a closed subset of X, thus it is compact. Then

the family C_i^c is an open cover of O^c, and thus there exist a finite subfamily $\mathcal{F} = \{C_{i_1}^c, \dots, C_{i_n}^c\}$ that covers O^c. In particular,

$$\bigcap_{C_{i_j}^c \in \mathcal{F}} C_{i_j} \subseteq O.$$

Ex.6:

For a space X, let X' be the subspace of X obtained by removing all the isolated points of X, i.e. all the points of X which are open and closed in X. Let B_n be the subspace of $[0,1]$ that consists of all the numbers having a base 2 decimal expansion $.a_1 a_2 \dots$ in which at most n of the digits a_i are 1, and let $B = \cup B_n$. Draw a picture of B_1 and B_2 and determine B'_n and B'. Deduce that there for each n there is a space X such that the sequence

$$X \supset X' \supset X'' \supset \cdots$$

becomes the empty set only after n stages.

Solution:
We start by describing B. By definition, the set B consists of all the elements that have a base 2 expansion that contains only finitely many 1. Let $x = .a_1 a_2 \dots \in B_n \subseteq B$ and let $i_1, \dots, i_m$, for $m \leq n$, be such that a_{i_j} is a one. Then

$$x = \frac{2^{i_m - i_1}}{2^{i_m}} + \frac{2^{i_m - i_2}}{2^{i_m}} + \cdots + \frac{1}{2^{i_m}}.$$

This means that the elements of B are all numbers of the form $\frac{p}{2^n}$, for every $n > 0$ and $p < 2^n$ integer. In particular, B does not have isolated points, and thus $B = B'$.

Consider B_n instead. A point x in B_n is isolated if and only if it contains exactly n 1-digits, that is, if and only if it belongs to $B_n - B_{n-1}$. Let $x = .a_1 a_2 \dots a_n$ be a point in B_n, and let $a_{i_1}, \dots, a_{i_m}$ be the 1-digits of x. If $m = n$, then every other point x' of B_n differs from x in at least one digit, and this digit cannot be after a_{i_m}. Indeed, otherwise the point x' will have the same n 1-digits of x plus at least one more, which is a contradiction. In particular, we can explicitly compute the minimal distance between x and every other point of B_n.

In the other case, let $x = a_1 \dots a_m$. Then the sequence y_n obtained adding n zero-digits after a_m and then a 1-digit shows that x is not isolated. Thus the non-isolated points of B_n are exactly the points in B_{n-1}.