**Topology** **Exercise Sheet 5**

Prof. Dr. Alessandro Sisto
Luca De Rosa

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The Exercise 7 is the hardest, and should be left as last one. All the exercises admit rather short solution.

# Ex.1:

Let $\mathcal{T}$ be the following topology on the real line $\mathbb{R}$:

- $\emptyset \in \mathcal{T}$;
- for each finite set $F \subset \mathbb{R}$, we declare $\mathbb{R} - F \in \mathcal{T}$.

(a) Check that $\mathcal{T}$ is a topology and that $(\mathbb{R}, \mathcal{T})$ is compact.

**Solution:**
For the fact that $\mathcal{T}$ is a topology:

(i) $\emptyset$ and $\mathbb{R}$ are contained in $\mathcal{T}$.

(ii) If $\mathbb{R} - F_1$ and $\mathbb{R} - F_2$ are contained in $\mathcal{T}$, then their intersection is $\mathbb{R} - (F_1 \cup F_2)$. Thus $\mathcal{T}$ is closed under finite intersections.

(iii) If $\mathbb{R} - F_i$ is a family of sets in $\mathcal{T}$, then $\bigcup(\mathbb{R} - F_i) = \mathbb{R} - \cap F_i$. Since the intersection of (arbitrarily many) finite sets is finite, we have that $\mathcal{T}$ is closed under union.

For the fact that $(\mathbb{R}, \mathcal{T})$ is compact. Let $\mathcal{O} = \{\mathbb{R} - F_i\}$ be an open cover. This means that $\bigcup(\mathbb{R} - F_i) = \mathbb{R}$, that is, $\bigcap F_i = \emptyset$. Consider $F_1$, and let $s = |F_1|$ be the cardinality of $F_1$. Thus, there are at most $s$ elements $F_{i_1}, \ldots, F_{i_s}$ such that $\left(\bigcap_{j=1}^{s} F_{i_j}\right) \cap F_1 = \emptyset$. Setting $F_1 = F_{i_0}$ we have that $\mathcal{O}' = \{\mathbb{R} - F_{i_j} \mid j = 0, \ldots, s\}$ is a finite subcover of $\mathcal{O}$.

(b) Let $\mathcal{T}_{\text{std}}$ be the standard topology on $\mathbb{R}$. Show that $(\mathbb{R}, \mathcal{T})$ and $(\mathbb{R}, \mathcal{T}_{\text{std}})$ are not homeomorphic.

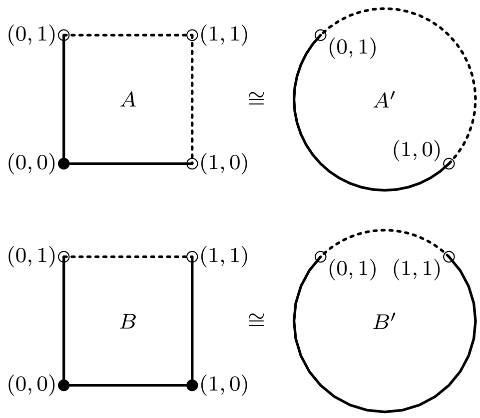
**Solution:**
If they where, then $(\mathbb{R}, \mathcal{T}_{\text{std}})$ would be compact. However, consider the following open cover: for each $z \in \mathbb{Z}$, let $O_z = \left(z - \frac{1}{2}, z + \frac{1}{2}\right)$, and let $\mathcal{O} = \{O_z \mid z \in \mathbb{Z}\}$. Since every $z \in \mathbb{Z}$ is contained only in the set $O_z$, if we remove any element from $\mathcal{O}$ we wouldn't have a cover anymore. Thus $(\mathbb{R}, \mathcal{T}_{\text{std}})$ is not compact and hence not homeomorphic to $(\mathbb{R}, \mathcal{T})$.

# Ex.2:

Show that $[0,1) \times [0,1)$ is homeomorphic to $[0,1] \times [0,1)$, but not to $[0,1] \times [0,1]$.

**Solution:**
Let $A = [0,1) \times [0,1)$, $B = [0,1] \times [0,1)$, and $C = [0,1] \times [0,1]$. We have that $C$ is compact (product of compact sets), but $A$ and $B$ are not. Indeed, they are not closed sets and a set in $\mathbb{R}^2$ is compact if and only if it is closed and bounded. Thus $A$ and $C$ are not homeomorphic.

It is possible to write an explicit homeomorphism between $A$ and $B$, but this would probably be just a very long formula that does not explain the ideas. For this reason, we will find homeomorphic spaces $A' \cong A$ and $B' \cong B$ and give an explicit formula only for $A' \cong B'$. Let $A'$ be the image of $A$ under the homeomorphism between the square $[0,1] \times [0,1]$ and the disk $\{(x,y) \in \mathbb{R}^2 \mid \sqrt{x^2+y^2} \leq 1\}$, and similarly $B'$ (see Exercise sheet 3, Question 3.iii).



If we look at $A'$ and $B'$ in polar coordinates, we have that the map $f\colon A' \to B'$ defined as follows is a homeomorphism:

$$f(\rho,\theta) = \begin{cases} (\rho,\theta) & \text{if } \theta \in \left[\frac{3}{4}\pi, \frac{5}{4}\pi\right]; \\ \left(\rho, \frac{5}{4}\pi + 2\left(\theta - \frac{5}{4}\pi\right)\right) = \left(\rho, 2\theta - \frac{5}{4}\pi\right) & \text{if } \theta \in \left[\frac{5}{4}\pi, \frac{7}{4}\pi\right]; \\ \left(\rho, \frac{3}{4}\pi - \frac{1}{2}\left(\frac{3}{4}\pi - \theta\right)\right) = \left(\rho, \frac{1}{2}\theta + \frac{3}{8}\pi\right) & \text{if } \theta \in \left[\frac{7}{4}\pi, \frac{3}{4}\pi\right]. \end{cases}$$

Intuitively, the map $f$ "drags" the point $(1,0)$ in $A'$ to the point $(1,1)$ in $B'$.

# Ex.3:

Let $X_i$, for $i \in I$, be a family of Hausdorff topological spaces. Show that $X = \prod_{i\in I} X_i$ is a Hausdorff space.

**Solution:**
Let $x = \{x_i\}_{i\in I}$, $y = \{y_i\}_{i\in I}$ be two distinct point of $X$. In particular, there is a coordinate $j$ such that $x_j \neq y_j$. Thus, we can find disjoint open sets $O_j$ and $U_j$ of $X_j$ such that $x_j \in O_j$ and $y_j \in U_j$. Let $O = \prod_{i\in I-\{j\}} X_j \times O_j$ and $U = \prod_{i\in I-\{j\}} X_i \times U_j$. Then $O$ and $U$ are disjoint open sets of $X$ such that $x \in O$ and $y \in U$.

# Ex.4:

Write down an example of a topological space that is not Hausdorff. *Note: recall that metric spaces are Hausdorff. If you have an example, check that is not a subspace of metric space (i.e. with respect to the induced topology).*

**Solution:**
Let $X$ be the set consisting of the points $\{a, b, c, d\}$. Let $\mathcal{T}$ be the following topology on $X$:

$$\mathcal{T} = \{\emptyset, \{a, b\}, \{c, d\}, X\}.$$

Every open set that contains $a$ contains also $b$. Thus $X$ is not Hausdorff.

## Ex.5:

Let $X$ be a first-countable topological space, $x$ be a point of $X$ and $\{O_\alpha\}_{\alpha=1}^{\infty}$ be a neighborhood basis for $x$.

(i) For each $n \in N$, let $U_n = \bigcap_{\alpha=1}^{n} O_\alpha$, and let $x_n$ be any point in $U_n$. Show that $\{x_n\}$ converges to $x$.

> **Solution:**
> Let $V$ be an open set that contains $x$. Since $\{O_\alpha\}$ is a neighborhood basis for $x$ there exists $\alpha \in \mathbb{N}$ such that $O_\alpha \subseteq V$. In particular, for each $m \geq \alpha$ we have that $U_m \subseteq V$. Hence there are at most $m$ points in the sequence $\{x_n\}$ that are not contained in $V$.

(ii) Let $\{y_i\}$ be a sequence such that for each $n \in \mathbb{N}$ and $\alpha \in \mathbb{N}$ there is an $i > n$ such that $y_i \in O_\alpha$. Show that there exists a subsequence $\{y_{i_j}\}$ of $\{y_i\}$ that converges to $x$.

> **Solution:**
> We define $y_{i_1}$ to be any point $y_s$ of the sequence that is contained in $O_1$, where $O_1$ is the first neighborhood of the basis $\{O_\alpha\}$. Now, suppose that the point $y_{i_m}$ was defined. We define $y_{i_m}$ as follows: let $U_m = \bigcap_{\alpha=1}^{m} O_\alpha$. Since $U_m$ is an open set that contains $x$, there is a neighborhood $O_{\alpha_m}$ of $x$ that is contained in $U_m$. By hypothesis, there is a point $y_s$ with $s > i_{m-1}$ such that $y_s \in O_{\alpha_m}$. We choose $y_{i_m} = y_s$. Then the exercise is concluded by part (i).

## Ex.6:

Let $(\mathbb{R}, \mathcal{T})$ be the real line equipped with the topology described in Question 1. Show that $(\mathbb{R}, \mathcal{T})$ is not first countable.

> **Solution:**
> Choose a point of $\mathbb{R}$, say 0, and suppose that there is a countable neighborhood basis $\{O_\alpha\}$ for 0. We will construct an open set $U$ such that every neighborhood $\{O_\alpha\}$ is not contained in $U$. We know that every neighborhood $\{O_\alpha\}$ has the

form $\mathbb{R} - F_\alpha$, where $F_\alpha$ is a finite set. Then $\mathcal{F} = \bigcup_{\alpha=1}^{\infty} F_\alpha$ is a countable union of finite sets, thus is countable. Since $\mathbb{R}$ is uncountable, there is a point $x$ (indeed, uncountably many) that is not contained in $\mathcal{F}$. Thus $U = \mathbb{R} - \{x\}$ is an open set that does not contain any $O_\alpha$, for $\alpha \in \mathbb{N}$.

# Ex.7:

Let $I = [0,1]$ and consider the space $I^I$ (that is, $I$-many copies of $I$). Show that $I^I$ is compact but not sequentially compact.

*Hints:*

*(i) You are allowed to use Tychonoff's theorem in the case of an uncountable product.*

*(ii) You may want to think of the space $I^I$ as the space of functions $f\colon I \to I$.*

*(iii) To check that a subsequence does not converge, it is enough to show that it does not converge on a coordinate.*

**Solution:**
Since $I^I$ is the product of compact sets, it is compact. We will show that is not sequentially compact.

For each $n \in \mathbb{N}$, let $f_n\colon I \to I$ be the function that associate to $x$ the $n$-th binary digit of $x$, and consider the sequence $\{f_n\}_{n\in\mathbb{N}}$. We claim that such a sequence does not have a converging subsequence. Suppose that there was a converging subsequence $\{f_{n_j}\}$. Then we can find a point $\overline{x} \in I$ such that the $n_1$-th binary digit of $\overline{x}$ is a 1, the $n_2$-th digit is a 0, the $n_3$-th is a 1 and so on. That is, we can find a coordinate $\overline{x} \in I$ such that the sequence $\{f_{n_j}\}$ restricted on $\overline{x}$ is 1,0,1,0,1,0,1,0, .... Thus $\{f_{n_j}\}$ do not converge.

Since the subsequence $\{f_{n_j}\}$ is arbitrary, we get the result.