Topology **Exercise Sheet 6**

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Exercise 2 is the most important. It consists of four parts, which may be considered as four independent exercises.

Ex.1:

(a) Let X be a metric space, $x \in X$ be a point of X and $\{O_\alpha\}$ be an open cover. Show that for every O_α that contains x there exists ε such that $B_\varepsilon(x) \subseteq O_\alpha$.

> **Solution:**
> We know that $B_{\frac{1}{n}}(x)$ is a neighbourhood basis for x. By definition, for every open set O such that $x \in O$, there is an element of the basis that is contained in O.

(b) Let X be a compact metric space and let $\{O_\alpha\}$ be an open cover for X. Show that there exists a Lebesgue number for the cover $\{O_\alpha\}$ using the fact that X is sequentially compact.

> **Solution:**
> Suppose that the cover $\{O_\alpha\}$ does not admit a Lebesgue number. We will show that this implies that there is a sequence with no converging subsequences, which is a contradiction.
>
> The fact that $\{O_\alpha\}$ does not admit a Lebesgue number means that for every $\varepsilon > 0$ there exists $x \in X$ such that for every $O_\alpha \in \{O_\alpha\}$ we have $B_\varepsilon(x) \nsubseteq O_\alpha$. Consider the sequence $\{x_n\}_{n\in\mathbb{N}}$ defined by choosing x_n to be the point that we find setting $\varepsilon = \frac{1}{n}$. We claim that such a sequence does not admit any convergent subsequence. Indeed, suppose that $\overline{x}$ was a limit point for a subsequence $\{x_{i_j}\}$, and let $O_\beta \in \{O_\alpha\}$ be such that $\overline{x} \in O_\beta$. By part (a), there exists $M > 0$ such that for all $m > M$ we have $B_{\frac{1}{m}}(\overline{x}) \subseteq O_\alpha$. Since $\overline{x}$ is a limit point for the subsequence $\{x_{i_j}\}$, every open set that contains $\overline{x}$ contains all but finitely many elements of $\{x_{i_j}\}$. In particular this holds true if we consider the open set $B_{\frac{1}{2m}}(\overline{x})$, for some $m > M$. Hence that there is $n > 2m$ such that $x_n \in B_{\frac{1}{2m}}(\overline{x})$. We recall that x_n has the property that for every element $O_\alpha \in \{O_\alpha\}$ we have $B_{\frac{1}{n}}(x_n) \nsubseteq O_\alpha$.

In particular, $B_{\frac{1}{n}}(x_n) \not\subseteq O_\beta$. But this is a contradiction. Indeed, since $x_n \in B_{\frac{1}{2m}}(\overline{x})$, and since $\frac{1}{n} < \frac{1}{2m}$, we have that $B_{\frac{1}{n}}(x_n) \subseteq B_{\frac{1}{m}}(\overline{x}) \subseteq O_\beta$.

Ex.2:

Let (X, d_X) be a compact metric space and (Y, d_Y) be a complete metric spaces, and let $C(X, Y) = \{f \colon X \to Y \mid f \text{ is continuous}\}$. We recall that the distance in $C(X, Y)$ is defined as $d(f, g) = \sup_{x \in X}\{d_Y(f(x), g(x))\}$.

(a) Show that d is indeed a metric.

Solution:

(i) $d(f, g) \geq 0$: This is because the distance is a sup of a non-empty set of non-negative values (because d_Y is a metric). Moreover, given f, g, we have that $d(f, g) < \infty$. Indeed, since X is compact we have that there is $x_0 \in X$ such that $d_Y(f(x_0), g(x_0)) = \sup_{x \in X}\{d_Y(f(x), g(x))\}$. Since the distance in Y between two points is always finite, we obtain that $d(f, g) < \infty$.

(ii) $d(f, g) = 0 \Leftrightarrow f = g$: It is clear that if $f = g$, then $d(f, g) = 0$. On the other hand, we have that $\sup_{x \in X}\{d(f(x), g(x))\} = \inf_{x \in X}\{d(f(x), g(x))\} = 0$. Thus for every $x \in X$ we have $d_Y(f(x), g(x)) = 0$. Since d_Y is a metric, we obtain $f(x) = g(x)$, for all $x \in X$.

(iii) $d(f, g) = d(g, f)$: This is clear since it holds for d_Y.

(iv) $d(f, g) + d(g, h) \geq d(f, h)$: For every $x \in X$, the triangle inequality holds on $f(x), g(x), h(x)$. Since distance are positive we have

$$\begin{aligned} d(f, g) + d(g, h) &= \sup_{x \in X}\{d_Y(f(x), g(x))\} + \sup_{x \in X}\{d_Y(g(x), h(x))\} = \\ &= \sup_{x \in X}\{d_Y(f(x), g(x)) + d_Y(g(x), h(x))\} \geq \\ &\geq \sup_{x \in X}\{d_Y(f(x), h(x))\} = d(f, h) \end{aligned}$$

(b) Show that $C(X, Y)$ is a complete metric space.

Solution:
Let $\{f_n\}$ be a Cauchy sequence in $C(X,Y)$. We have that for each $x \in X$, the sequence $\{f_n(x)\}$ is a Cauchy sequence in Y. Thus it converges to a point $\overline{f(x)}$. We claim that the function f defined as $f(x) = \overline{f(x)}$ is the limit of the sequence $\{f_n\}$ in $C(X,Y)$. Since X and Y are first-countable, we saw in the lecture that such an f is continuous. Fix $\varepsilon > 0$. We want to show that there are only finitely many n such that $d_{C(X,Y)}(f_n, f) > \varepsilon$. Since $\{f_n\}$ is a Cauchy sequence, there is $M > 0$ such that for every $n_1, n_2 \geq M$, it holds that $d_{C(X,Y)}(f_{n_1}, f_{n_2}) < \frac{\varepsilon}{2}$. In particular, this holds true for every point $x \in X$, that is $d_Y(f_{n_1}(x), f_{n_2}(x)) < \frac{\varepsilon}{2}$. Choosing n_2 large enough, we have that $d_Y(\overline{f(x)}, f_{n_2}(x)) \leq \frac{1}{2}\varepsilon$, thus $d(f_{n_1}(x), f(x)) \leq \varepsilon$. Note that n_2 may depend on x, but n_1 does not. Thus f is the limit of the sequence.

(c) Let $\mathcal{F} \subseteq C(X,Y)$ be a compact set. Show that $\mathcal{F}$ is closed and equicontinuous.

Solution:
We recall that a subset $\mathcal{F}$ of $C(X,Y)$ is equicontinuous if for every ε there exists δ such that for every $x \in X$ and $f \in \mathcal{F}$, if $d_X(x,y) < \delta$, then $d_Y(f(x), f(y)) < \varepsilon$. We know that a compact set in a metric space is complete and bounded. In particular, complete implies closed. We need to show that $\mathcal{F}$ is equicontinuous. Fix $\varepsilon > 0$. We want to find δ that satisfies the equicontinuity condition. Let $r = \frac{1}{4}\varepsilon$ and consider the open cover $\{B_r(f)\}_{f \in \mathcal{F}}$. Since $\mathcal{F}$ is compact, there is a finite subcover $\{B_r(f_1), \ldots, B_r(f_n)\}$. Moreover, since X is compact, all elements $f \in \mathcal{F}$ (and in particular the f_i) are uniformly continuous. This means that for every function f_i, there exists δ_i such that, for every $x \in X$, if $d(x,y) < \delta_i$, then $d(f(x), f(y)) < \frac{1}{2}\varepsilon$. Let $\delta = \min\{\delta_i\}$. Since there are finitely many functions f_i we have that the minimum is defined and $\delta > 0$. We claim that δ is the desired constant. Indeed, let $f \in \mathcal{F}$. Then there is f_i such that $d(f, f_i) < r$. This means that for every $x \in X$, we have that $d(f(x), f_i(x)) \leq r = \frac{1}{4}\varepsilon$. Let $y \in X$ such that $d(x,y) \leq \delta$. Then $d(f_i(x), f_i(y)) < \frac{1}{2}\varepsilon$. By triangular inequality:

$$d(f(x), f(y)) \leq d(f(x), f_i(x)) + d(f_i(x), f_i(y)) + d(f_i(y), f(y)) <$$
$$< \frac{1}{4}\varepsilon + \frac{1}{2}\varepsilon + \frac{1}{4}\varepsilon = \varepsilon.$$

(d) Let $\mathcal{F}$ be a subset of $C(X,Y)$. Show that $\mathcal{F}$ is equicontinuous if and only if

$\overline{\mathcal{F}}$ is.

Solution:
Suppose that $\mathcal{F}$ is equicontinuous, that is for every ε there exists δ such that for every $x \in X$ and $f \in \mathcal{F}$, if $d_X(x, y) < \delta$, then $d_Y(f(x), f(y)) < \varepsilon$. We will show that this holds true also for every $f \in \overline{\mathcal{F}}$ (with the same constants). Indeed, fix ε and let $f \in \overline{\mathcal{F}}$. Then there exists a sequence $\{f_n\}$ of functions in $\mathcal{F}$ that converges to f. Up to possibly passing to a subsequence we can assume that for every $x \in X$ we have $d(f(x), f_n(x)) \leq \frac{1}{n}$. Since $\mathcal{F}$ is equicontinuous, we we have that for every y such that $d(x, y) \leq \delta$, we have that $d(f_n(x), f_n(y)) \leq \varepsilon$. By triangular inequality:

$$d(f(x), f(y)) \leq d(f(x), f_n(x)) + d(f_n(x), f_n(y)) + d(f_n(y), f(y)) \leq \varepsilon + \frac{2}{n}.$$

However, n can be chosen arbitrarily large, thus $d(f(x), f(y)) \leq \varepsilon$.

On the other hand if $\overline{\mathcal{F}}$ is equicontinuous, so is $\mathcal{F}$, since every element of $\mathcal{F}$ is an element of $\overline{\mathcal{F}}$.

Ex.3:

Let Z be a complete metric space and let Y be a subset of Z. Show that Y is complete if and only if it is closed.

Solution:
We will use the Statement 3 of Lecture 10 (see the List of statements on the website). Suppose that Y is not complete. Then there is a Cauchy sequence $\{x_i\}$, with $x_i \subseteq Y$ that does not admit a limit in Y. Since Z is complete, $\{x_i\}$ admits a limit $x \in Z$. Thus $x \in Z - Y$ and $x \in \overline{Y}$, which means that Y is not closed in Z.

On the other hand, suppose that Y is complete. Thus every Cauchy sequence admits a limit in Y. However, each converging sequence in a metric space is a Cauchy sequence. Thus $Y = \overline{Y}$, and hence is closed.

Ex.4:

Let X be a compact topological space, and consider the space $C(X, \mathbb{R})$. Show that $C(X, \mathbb{R})$ is a ring.

Solution:

Given two functions f and g we define $(f+g)(x) = f(x)+g(x)$ and $(f \cdot g)(x) = f(x)g(x)$. The 0-element is the constant function 0, the inverse of a function f is the function $-f$ defined as $(-f)(x) = -f(x)$. Finally, the 1-element is the constant function 1. Associativity of both the operations, commutativity of the sum and distributivity are guaranteed by the fact that $\mathbb{R}$ is a ring.