Topology **Exercise Sheet 7**

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Ex.1:

Let X and Y be topological spaces. Suppose that Y is Hausdorff and that there is a continuous bijection $f \colon X \to Y$. Show that X is Hausdorff.

Solution:
Let x_1, x_2 be distinct points of X. We want to find disjoint disjoint open sets O_1 and O_2 such that $x_i \in O_i$. Since f is a bijection, $f(x_1)$ and $f(x_2)$ are distinct. Since Y is Hausdorff, there are disjoint open subsets of Y U_1 and U_2 such that $f(x_i) \in O_i$. Since f is continuous, $f^{-1}(U_1)$ and $f^{-1}(U_2)$ are disjoint open sets. Setting $O_i = f^{-1}(U_i)$ provides the result.

Ex.2:

Let X be a Hausdorff topological space and let Y be a subset of X. Show that Y is Hausdorff with respect to the induced topology.

Solution:
Let $y_1 \neq y_2$ be points of Y (and thus of X). Since X is Hausdorff, there are disjoint open sets O_1 and O_2 such that $y_i \in O_i$. Thus, $Y \cap O_1$ and $Y \cap O_2$ are disjoint open sets of Y that contain y_1 and y_2.

Ex.3:

Consider the following two facts:

1. A homeomorphism between locally compact Hausdorff spaces extends to a homeomorphism between the one-point compactifications. In other words, homeomorphic locally compact Hausdorff spaces have homeomorphic one-point compactifications.

2. Given a point $p \in S^n$, the one point compactification of $S^n \setminus \{p\}$ is S^n.

Use those facts to prove that the one-point compactification of $\mathbb{R}$ is homeomorphic to the 1-dimensional sphere $S^1 \subseteq \mathbb{R}^2$.

Can you generalise such homeomorphism to the n-dimensional case (i.e. write down the homeomorphism between $\mathbb{R}^n$ and $S^n \setminus \{p\}$)?

Solution:
Let $p = (0, 0, \ldots, 1)$ the "north pole" point. The punctured sphere $S^n \setminus \{p\}$ is homeomorphic to $\mathbb{R}^n$ via the homeomorphism

$$\phi : S^n \setminus \{p\} \longrightarrow \mathbb{R}^n : (x_1, x_2, \ldots, x_{n+1}) \mapsto \frac{1}{1 - x_{n+1}}(x_1, \ldots, x_n)$$

By the above two facts, such map extends to the homeomorphism we are looking for.

Ex.4:

A map $f : X \to Y$ is called *proper* if for every compact $K \subseteq Y$ we have that $f^{-1}(K)$ is compact. Let $f : X \to Y$ be a proper, continuous map. Show that f extends to a continuous map $\widehat{f} : \widehat{X} \to \widehat{Y}$.

Solution:
Denoting with ω_1 and ω_2 the points added to X and Y, respectively, we define $\widehat{f}(\omega_1) = \omega_2$, and $\widehat{f}(x) = f(x)$ for all $x \in X$. We need to show that for every open set $U \subseteq \widehat{Y}$, the preimage of U is open. If $U \subseteq Y$, then $\widehat{f}^{-1}(U) = f^{-1}(U)$, which is open. Otherwise, $U = \{\omega_2\} \cup (Y - K)$, where K is compact. Thus, $f^{-1}(U) = f^{-1}(Y - K) = X - f^{-1}(K)$. Since f is proper, $f^{-1}(K)$ is compact in X, thus $f^{-1}(U)$ is open.

Ex.5:

Given the statement *'Every discrete subset of a compact set is finite'*, argue if it is true or not. If yes, prove it, if no, find a counterexample.

We recall that a subset Y of a topological space is discrete if every point $y \in Y$ is open in the subset topology (note that this implies that every point of Y is open and closed in the subset topology).

Solution:
It is false. An example is given by the sequence $\left\{\frac{1}{n}\right\}$ in the interval $[0, 1]$. However, the statement is correct if we ask for a closed discrete subset. Indeed, let X be compact and Y be a closed discrete subset. Then for every $y \in Y$ there is an open U_y of X such that $U_y \cap Y = \{y\}$. Thus the set $\{U_y \mid y \in Y\} \cup \{X - Y\}$ is an open cover of X. However, the statement is true if we consider discrete closed sets. Indeed, if $Y \subseteq X$ is discrete, for each $y \in Y$ there exists an open set U_y of X such that $U_y \cap Y = \{y\}$. Since Y is closed in X, the set Y^c is open in X. Thus $\mathcal{C} = \{U_y \mid y \in Y\} \cup \{Y^c\}$ is an open cover of X. Note that we need to add $\{Y^c\}$ to be sure that it is a cover of X, and in order to do so we need Y to be closed.

Then there exists a finite subcover. However, since each point of Y is contained in exactly one element of $\mathcal{C}$ we have that Y has to be finite.