Topology **Exercise Sheet 8**

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With this exercise sheet we want you to get more used to quotient topology. Ex.1 provides a useful source of examples and counterexamples for exercise 2, so they go together. Ex.3 is relatively easy and short. Ex.4 relates connected components and quotient topology. Finally, consider Ex.5 as an extra optional technical training.

Ex.1:

Consider the following equivalence relation on $\mathbb{R}$:

$$x \sim y \Leftrightarrow \begin{cases} x = y & \text{or} \\ |x| = |y| & \text{and } |x| > 1. \end{cases}$$

Let $X = \mathbb{R}/\sim$ equipped with the induce topology. Show that X is not an Hausdorff space and that every point of X has a neighborhood homeomorphic to the interval $(-1, 1)$.

Solution:
Let $q\colon \mathbb{R} \to X$ be the quotient map. Note that $q(1) \neq q(-1)$. Let U and V be open sets of X such that $q(1) \in U$ and $q(-1) \in V$. We claim that U and V must intersect. Consider $q^{-1}(U)$ and $q^{-1}(V)$. They are open sets of $\mathbb{R}$ that contain 1 and -1 respectively. In particular, there are ε_1 and ε_2 such that $(1 - \varepsilon_1, 1 + \varepsilon - 1) \subseteq q^{-1}(U)$ and $(-1 - \varepsilon_2, -1 + \varepsilon_2 \subseteq q^{-1}(V)$. In particular, if $\varepsilon < \min\{\varepsilon_1, \varepsilon_2\}$ (note that such an ε always exists), we have that $1+\varepsilon \in q^{-1}(U)$ and $-1 - \varepsilon \in q^{-1}(V)$. However, since $q(1 + \varepsilon) = q(1 - \varepsilon)$, we have that U and V intersects, which concludes the first part of the exercise.

For the second part, let x be a point of X, and suppose that $x = q(y)$ for some $y \in \mathbb{R}$. We claim that $q((y - 1, y + 1)$ is the desired neighbourhood: in particular we will show that $(y-1, y+1)$ and $q((y-1, y+1))$ are homeomorphic. It is clear that $(y - 1, y + 1)$ and $(-1, 1)$ are homeomorphic.

To simplify notation, let $I = (y - 1, y + 1)$. Note that there do not exist two distinct points $z_1, z_2 \in I$ such that $|z_1| = |z_2| > 1$. This implies that $q_{|I}$ is injective. This means that $q_{|I}\colon I \to q(I)$ is a bijection. We recall that if $Y \subseteq Z$

is an open subset of a topological space, a subset $V \subseteq Y$ is open in Y (with respect of the induced topology) if and only if it is open in Z. Note that this is not true if Y is closed. Example: $[0, 1)$ is an open subset of $[0, 2)$, but is not an open subset of $\mathbb{R}$. Note that I, $q(I)$ are open in their ambient spaces, thus the above applies.

This (actually for the next implication we only need $q(I)$ to be open) implies that $q_{|I}$ is continuous. Indeed a set $V \subseteq q(I)$ is open if it is open in X. Since q is continuous, then $q^{-1}(V)$ is open, so it is $q^{-1}(V) \cap I$ in the induced topology.

We claim that q is open. Note that, since I is an open set, this would imply that $q_{|I}$ is open, which is the last thing that we need to prove. Let U be open in $\mathbb{R}$. We need to show that $q^{-1}(q(U))$ is open. By definition of quotient topology, this implies that $q(U)$ is open. We can assume that U is an interval: recall that an open in $\mathbb{R}$ is the disjoint union of intervals. Then apply the reasoning below to each of those intervals.

If $U \subseteq [-1, 1]$, then $q^{-1}(q(U)) = U$, thus we are done. If $U = (a, b)$, with $a > 1$ or $b < 1$, then $q^{-1}(q(U)) = (-b, -a) \cup (a, b)$. So suppose that $U = (a, b)$ with $-1 < a < 1 < b$ (that is, $1 \in U$. Then $q^{-1}(q(U)) = (-b, -1) \cup (a, b)$. We obtain a similar result if $-1 \in U$. Finally, if both 1 and -1 are contained in U, then subdivide $U = (a, \frac{1}{2}) \cup (-\frac{1}{2}, b)$ and apply the reasoning above.

Ex.2:

Let X, Y be topological spaces, and let $f\colon X \to Y$ be a continuous surjection. Assume that Y is equipped with the quotient topology, which means that a set $U \subseteq Y$ is open if an only if $f^{-1}(U)$ is open in X. Decide if the following are true or false: in case they are true, prove them, in case they are false, find a counterexample.

(a) If X is compact, so it is Y;

Solution:
This is true, because Y is the continuous image of a compact set.

(b) If X is Hausdorff, so it is Y;

Solution:
This is false. The space X of Question 1 is not Hausdorff, even though $\mathbb{R}$ is.

(c) If X is normal, then Y is Hausdorff;

Solution:
This is false. Again the space X of Question 1 is not Hausdorff (hence not normal) and the quotient of a normal space.

(d) If $|X| = \infty$, then $|Y| = \infty$;

Solution:
This is false. For any X, the map $f\colon X \to \{*\}$, where $*$ is a single point, is continuous. Moreover, $f^{-1}(\{*\}) = X$, which show that the requirement of the exercise are met. Since there are no assumption on X, we can just pick any space with infinitely may points and map it to a single point.

(e) If X is connected, so it is Y;

Solution:
This is true, because Y is the continuous image of a connected set. We recall the proof of why the continuous image of a connected set is connected. Suppose that Y was not connected. Then $Y = Y_1 \sqcup Y_2$, with Y_i open. Thus $X = f^{-1}(Y_1) \sqcup f^{-1}(Y_2)$, which is a contradiction.

(f) If X is a metric space, so it is Y.

Solution:
This is false. Since metric spaces are Hausdorff, the space of Question 1 furnishes a counterexample.

Ex.3:

Prove or disprove with a counterexample the following statement: *Let X be a compact space, and let $q\colon X \to Y$ be a quotient of X. Then the map q is open.*

Hint: You may want to consider the interval $[-2,2]$ with all the points in $[-1,1]$ identified.

Solution:
This is false. Consider the interval $[-2,2]$, and take the quotient by the equivalence relation $x \sim y \Leftrightarrow x \in [-1,1]$ and $y \in [-1,1]$. Let q be the quotient map. Then $q^{-1}(q((0,2))) = [-1,2)$, which is not an open set. Thus $q((0,2))$ is not open in the quotient, even if $(0,2)$ is open in $[-2,2]$.

Ex.4:

Let X be a topological space, and assume that all connected components of X are open. (Note, this is not always true: example $\mathbb{Q} \subseteq \mathbb{R}$ with the induced topology.) Let $q\colon X \to Y$ be a quotient of X. Show that the connected components of Y are also open.

Solution:
Let C be a connected component of Y. We claim that $q^{-1}(Y)$ is the union of connected components of X. Note that in general we cannot expect $q^{-1}(C)$ to be a single connected component. The claim is saying that if $q^{-1}(C)$ intersects a connected component K of X, then $q^{-1}(C)$ has to contain the whole K. Indeed, suppose that this was not the case: then $q(K) \cap C \neq \emptyset$. Moreover, $q(K)$ is connected, thus $q(K) \cup C$ is a connected set that is strictly larger than C, which is a contradiction.

Thus $q^{-1}(C) = \bigcup K_i$, where each K_i is a connected component of X. Since, by assumption, all the K_i are open, we obtain that $q^{-1}(C)$ is open. By the definition of quotient topology we have that C is open in Y.

Ex.5:

Let $q\colon X \to Y$ be a quotient and let $A \subset X$ be such that $q^{-1}(q(A)) = A$. A set A with this property is called *saturated*. Show that if q is an open map, then also $\overline{A}$ and $\mathrm{Int}(A)$ are saturated. Give an example where the map q is not open and the above is false (i.e. there is a saturated set B such that $\overline{B}$ or $\text{Int}(B)$ are not saturated).

Solution:
We start by showing that $\text{Int}(A)$ is saturated. Since A is saturated we have $(\text{Int}(A) \subseteq) q^{-1}(q(\text{Int}(A))) \subseteq A$. However, since q is open, we have that $q(\text{Int}(A))$ is open. Since Y is a quotient, $q(\text{Int}(A))$ is open if and only if $q^{-1}(q(\text{Int}(A)))$ is open. Since $\text{Int}(A)$ is the largest open that is contained in A, we get $\text{Int}(A) = q^{-1}(q(\text{Int}(A)))$.

For the closure, observe that the fact that A is saturated implies that $X - A$ is saturated. Thus $\text{Int}(X - A)$ is saturated, which implies that $X - \text{Int}(X - A) = \overline{A}$ is also saturated.

For the counterexample (which is also another one for Question 3), consider the quotient of the interval $[-2, 2]$ by the equivalence relation $-1 \sim 1$. Let $A = (0, 1)$. Then A is saturated, but $[0, 1]$ is not.