Prof. Dr. Alessandro Sisto
Luca De Rosa
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Ex.1:

Let X be a topological space, $q\colon X \to Y$ be the quotient map and let $f\colon Y \to Z$ be any function. Then f is continuous if and only if $f \circ q$ is continuous.

$$\begin{array}{ccc} X & & \\ \downarrow{\scriptstyle q} & \searrow^{q\circ f} & \\ Y & \xrightarrow{f} & Z \end{array}$$

Solution:
Since q is continuous, it is cleat that if f is continuous, so is $f \circ q$. So suppose that $f \circ q$ is continuous, and let $U \subseteq Z$. We want to show that $f^{-1}(U)$ is open. But since $q^{-1}(f^{-1}(U))$ is open, this is true by definition of quotient topology.

Ex.2:

Let X be a Hausdorff space and let K be a non-open compact in X. Show that the quotient X/K is Hausdorff.

Hint: It can be helpful to write down explicitly how the elements $[x] \in X/K$ look like, distinguishing the case $x \in K$ or not.

Solution:
Let $q\colon X \to X/K$ be the quotient map. The elements of X/K are equivalence classes $[x]$ such that $[x] = \{x\}$ if $x \notin K$ and $[x] = K$ otherwise. Let $[x] \neq [y]$ be distinct points of X/K. There are two cases: either both x and y are points of $X - K$, or (up to exchange x and y), $[y] = K$. In the first case, since X is Hausdorff there exist two disjoint open sets U, V such that $x \in U$ and $y \in V$. We want to modify U and V such that they do not intersect K. Note that, since X is Hausdorff, K is closed. Thus $X - K$ is open (and contains x, y). This means that, up to replace U with $U \cap (X - K)$ and similarly for V, we can assume that U and V are disjoint open sets contained in $X - K$. We claim that $q(U)$ and $q(Y)$ are disjoint open sets that contain $[x]$ and $[y]$ respectively.

It is clear that $[x] \in q(U)$, $[y] \in q(U)$ and that $q(U) \cap q(V) = \emptyset$. Moreover, by the definition of the quotient map, $q^{-1}(q(U)) = U$ and similarly for V. Thus, by definition of quotient topology, $q(U)$ and $q(V)$ are open.

Consider now the case $[y] = K$. If we can find disjoint open sets U and V such that $x \in U$ and $K \subseteq V$, then the same reasoning as before gives that $q(U)$ and $q(V)$ are disjoint open sets containing $[x], [y]$ respectively. For each point $k \in K$, let U_k and V_k be disjoint open sets such that $x \in U_k$ and $k \in V_k$. The existence of such open sets is guaranteed by the fact that X is Hausdorff. Note that $\{V_k \cap K\}$ is an open cover for K, thus it admits a finite subcover. Let $\{V_h\}_{h\in H}$ be such a subcover. Set $U = \bigcap_{h\in H} U_h$, $V = \bigcup_{h\in H} V_h$. Since H is finite, both V and U are open. Moreover, it is clear by construction that U and V are disjoint, which concludes the proof.

Ex.3:

Let X be a topological space, and let Δ be the diagonal of $X \times X$, i.e. the set $\Delta = \{(x, y) \in X \times X \mid x = y\}$. Show that X is Hausdorff if and only in Δ is closed in $X \times X$.

Solution:

Assume that X is Hausdorff, we will show that $Y = (X \times X) - \Delta$ is open. Let $(x, y) \in Y$, that is $x \neq y$. Then there exists disjoint open sets U, V such that $x \in U$ and $y \in V$. Then $U \times V$ is an open of $X \times X$ which contains (x, y). We claim that is contained in Y. Indeed, suppose that there was $z \in X$ such that $(z, z) \in U \times V$. This implies that $z \in U \cap V$, which is a contradiction.

On the other hand, assume that the space Y above is open, and let $(x, y) \in Y$. We want to find disjoint open sets U and V as before. Since Y is open, there is an open set O contained in Y that contains (x, y). By the definition of product topology, O is the union of products of the form $U_i \times V_i$, where U_i and V_i are open in X. In particular, there exists j such that $(x, y) \in U_j \times V_j \subseteq O \subseteq Y$. Then, as before, U_j and V_j are the desired open sets.

Ex.4:

Let X be Hausdorff, and let $\sim$ be an equivalence relation on X. Let $R = \{(x, y) \in X \times X \mid x \sim y\}$. Suppose that $p\colon X \to X/_\sim$ is open. Show that $X/_\sim$ is Hausdorff if and only if R is closed in $X \times X$.

Solution:
We start with the easier part: assume that $X/_\sim$ is Hausdorff, and let $(x, y) \in (X \times X) - R$. By definition of R we have that $p(x) \neq p(y)$. Thus we can find disjoint open sets U_x, U_y in $X_\sim$ that contain $p(x)$ and $p(y)$ respectively. By definition of quotient topology, $p^{-1}(U_x)$ and $p^{-1}(U_y)$ are open in X. Moreover, $p^{-1}(U_x) \times p^{-1}(U_y) \subseteq X \times X - R$. Indeed, suppose that there was a point $(z, t) \in R$ with $z \in p^{-1}(U_x)$ and $t \in p^{-1}(U_y)$. Then $z \sim t$ which contradicts U and V being disjoint in $X/_\sim$.

Now, let's consider the first implication. By Exercise Sheet 2, Question 6.a we have that $p \times p$ is open. By hypothesis, $X \times X - R$ is open. This implies that $p(X \times X - R)$ is open in $X/_\sim \times X/_\sim$. We claim that $p(X \times X - R) = X/_\sim \times X/_\sim - \Delta$, where Δ is the diagonal of $X/_\sim$. Note that then the exercise is concluded by Question 3.

Ex.5:

Show that there is a quotient map $q\colon (-2, 2) \to [-1, 1]$, but not a quotient map $p\colon [-2, 2] \to (-1, 1)$ (This means that there is a quotient of $(-2, 2)$ that is homeomorphic to $[-1, 1]$).

Solution:
Let $K = (-2, 1] \cup [1, 2)$. Then $(-2, 2)/K$ is homeomorphic to $[-1, 1]$. On the other hand, if there was a continuous surjection $p\colon [-2, 2] \to (-1, 1)$, we would have that the latter is compact, which is a contradiction.