Probability and Statistics

Exercise sheet 10

Exercise 10.1 Let X and Y be independent random variables such that $X \sim \text{Poi}(\lambda)$ and $Y \sim \text{Poi}(\mu)$. Show that given $X + Y = s$, the conditional distribution of X is $\text{Bin}\left(s, \frac{\lambda}{\lambda+\mu}\right)$.

Exercise 10.2 Let X, Y and Z be $\overset{\text{iid}}{\sim} \text{Exp}(1)$.

(a) Find $E[X\sqrt{X+Y}]$.

(b) Find $P(X < 2Y < 3Z)$.

Exercise 10.3 Suppose $X \sim \mathcal{N}(0, 1)$ and $Y \mid X = x \sim \mathcal{N}(x + 1, 1)$.

(a) What is the marginal distribution of Y?

(b) Find $\text{cov}(X, Y)$ and the correlation of X and Y.

(c) Find the conditional distribution of X given $Y = y$.

Exercise 10.4 Gaussian (normal) vectors.

A vector $X = (X_1, ..., X_n)^T$ is said to be a Gaussian vector if there is a matrix $A \in \mathbb{R}^{n\times n}$, a vector $Z = (Z_1, ..., Z_n)^T$ with $Z_1, ..., Z_n \overset{\text{iid}}{\sim} \mathcal{N}(0, 1)$ and $\mu \in \mathbb{R}^n$ such that

$$X \overset{\text{d}}{=} \mu + AZ. \tag{1}$$

In this case, note that $E(X) = \mu$ and $\text{var}(X) = AA^T =: \Sigma$. Here, the covariance matrix Σ is not necessarily invertible. If it is, then X admits a density with respect to Lebesgue measure on $\mathbb{R}^n$.

(a) Show that if X is a Gaussian vector, then any linear combination of $X_1, ..., X_n$ is a normal random variable.

(b) We want to show that the condition in (a) is also sufficient: i.e., that if any linear combination of $X_1, ..., X_n$ is a normal random variable, then X is a Gaussian vector.

1. Explain why you can write $\Sigma = P^T DP$ with P orthogonal and D a diagonal matrix with entries $\lambda_1, ..., \lambda_n \geq 0$. Let $A := P^T D^{\frac{1}{2}}$, where $D^{\frac{1}{2}}$ is defined as the diagonal matrix with entries $\lambda_1^{\frac{1}{2}}, ..., \lambda_n^{\frac{1}{2}}$.
2. For a fixed $v \in \mathbb{R}^n$, find the distribution of $v^T(\mu + AZ)$ with $Z = (Z_1, ..., Z_n)^T$ with $Z_1, ..., Z_n \overset{\text{iid}}{\sim} \mathcal{N}(0, 1)$.
   Hint: From a previous exercise sheet, you know that a linear combination of i.i.d standard Gaussians has a Gaussian distribution $\mathcal{N}(a, b^2)$. What are the parameters a, b^2 here?
3. What is the distribution of $v^T X$?
4. To conclude, use that fact that two random vectors W_1, W_2 in $\mathbb{R}^n$ have the same distribution if and only if

$$v^T W_1 \overset{\text{d}}{=} v^T W_2 \quad \forall v \in \mathbb{R}^n.$$

(c) Let $X \sim \mathcal{N}(0,1)$ and Z be a discrete random variable such that $X \perp\!\!\!\perp Z$ and $P(Z=-1) = P(Z=1) = \frac{1}{2}$.

Consider the random variable Y, defined by

$$Y = \begin{cases} X, & \text{if } Z = 1 \\ -X, & \text{if } Z = -1. \end{cases}$$

- Show that marginally, $Y \sim \mathcal{N}(0,1)$.
- Find $P(X+Y=0)$.
- Is (X,Y) a Gaussian vector? What do you conclude from this exercise?

Exercise 10.5 (optional).

The goal of this exercise is to manipulate the Jacobian formula to obtain the density of a convolution.

Let X and Y be two independent random variables with density f_X and f_Y respectively. We are interested in deriving the density of the random variable $X+Y$.

(a) Let $S := X+Y$, $T := Y$ and consider the map

$$\begin{aligned} g &: \mathbb{R}^2 \to \mathbb{R}^2 \\ (x,y) &\mapsto (s,t) = (x+y, y). \end{aligned}$$

Show that g is bijective with Jacobian $\neq 0$ at any $(x,y) \in \mathbb{R}^2$. Conclude that the random pair (S,T) has point density $f_{(S,T)}$ given by

$$f_{(S,T)}(s,t) = f_X(s-t) f_Y(t).$$

(b) Conclude that

$$f_S(s) = \int_{\mathbb{R}} f_X(s-y) f_Y(y) dy.$$

(c) Give the density of the convolution $X+Y$ in the following cases:

- $X \sim \text{Exp}(\lambda)$ and $Y \sim \text{Exp}(\mu)$. Do you recognise the distribution when $\mu = \lambda$?
- $X \sim G(\alpha_1, \beta)$ and $G(\alpha_2, \beta)$.
- $X \sim \mathcal{N}(0, \sigma_1^2)$ and $Y \sim \mathcal{N}(0, \sigma_2^2)$.