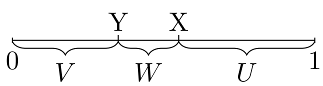
# Probability and Statistics

## Exercise sheet 11

**Exercise 11.1** (Breaking a stick) Suppose $X \sim \mathrm{U}([0,1])$ and $Y \mid X \sim \mathrm{U}([0,X])$. Consider now $U = 1-X, V = Y, W = X - Y$ (this represents breaking a stick into parts with length $X$ and $U$, and then breaking the left piece again into $V$ and $W$). Find $E[\max(U,V,W)]$.




**Exercise 11.2** (Uniforms, uniforms...) Suppose $X \sim U([0,1])$ and consider $Y = 2X$.

(a) What is the joint distribution of $(X,Y)$?

(b) Does this joint distribution have a density with respect to the Lebesgue measure on $\mathbb{R}^2$?

(c) (Probability of a diamond) Let $X, Y$ and $Z$ be $\overset{iid}{\sim} \mathrm{U}([-1,1])$. Find $P(|X|+|Y|+|Z| \leq 1)$.

**Exercise 11.3** Recall that $X = \begin{pmatrix} X_1 \\ \vdots \\ X_n \end{pmatrix} \sim \mathcal{N}(\mu, \Sigma)$ (a Gaussian vector with expectation $\mu = \begin{pmatrix} \mu_1 \\ \vdots \\ \mu_n \end{pmatrix} \in \mathbb{R}^n$ and covariance matrix $\Sigma \in \mathbb{R}^{n\times n}$) if for any $v \in \mathbb{R}^n$,

$$v^T X = \sum_{i=1}^{n} v_i X_i \sim \mathcal{N}(v^T\mu, v^T\Sigma v).$$

The goal of this exercise is to show the following remarkable property:
($*$) $X_1, ..., X_n$ are independent if and only if for all $i \neq j$, $\Sigma_{ij} = \mathrm{cov}(X_i, X_j) = 0$.

(a) Show that ($*$) is necessary.

(b) To show it is sufficient, we shall use the following result:
$X_1, ..., X_n$ are independent if and only if

$$\Psi_X(t) := E(e^{t^T X}) = \prod_{i=1}^{n} \Psi_{X_i}(t_i) \left( = \prod_{i=1}^{n} E(e^{t_i X_i}) \right)$$

for all $t \in \mathbb{R}^n$.
Compute $\Psi_X(t)$ when $\Sigma_{ij} = 0$ for all $i \neq j$ and conclude.
*Hint:* $t^T X$ is a normal random variable, for which we know the expression of the moment generating function.

(c) Taking $n \geq 3$, let $Y \in \mathbb{R}^p$ (for $2 \leq p \leq n-1$) be a subset of the original vector $X$. Using a simple argument, explain why $Y$ is also a Gaussian vector. When are the components of $Y$ independent?

(d) Let $X = \begin{pmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \\ X_5 \end{pmatrix} \in \mathbb{R}^5$ have a $\mathcal{N}(\mu, \Sigma)$ distribution, where

$$\mu = \begin{pmatrix} -1 \\ 2 \\ 0 \\ \frac{1}{2} \\ 3 \end{pmatrix}, \quad \Sigma = \begin{pmatrix} 9 & 0 & 0 & 1 & 2 \\ 0 & 2 & 0 & -1 & 6 \\ 0 & 0 & 16 & 0 & 3 \\ 1 & -1 & 0 & 4 & 3 \\ 2 & 6 & 3 & 3 & 49 \end{pmatrix}.$$

Which subsets of $X_1, ..., X_5$ can you say are independent?

(e) Consider the case $n = 2$, and $(X_1, X_2)^T$ a Gaussian pair with expectation $\mu = (\mu_1, \mu_2)^T$ and a $2 \times 2$ covariance matrix

$$\Sigma = \begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix}$$

where $\sigma_1^2, \sigma_2^2 > 0$ are the (marginal) variances and $\rho$ is the correlation.

Find $a$ and $b$ such that $X_1 + X_2$ and $aX_1 + bX_2$ are independent.

**Exercise 11.4** (I hope you're arriving soon)

Alice and Viera plan to meet at a café, and each will arrive at a random time between 15:00 and 15:30, independently of each other. Find the probability that the first to arrive has to wait between 5 and 10 minutes for the other to arrive.

**Exercise 11.5** (Cauchy-Schwarz) (optional)

The goal of this exercise is to show the Cauchy-Schwarz inequality, which is stated as follows.

Let $X$ and $Y$ be two random variables defined on the same probability space, such that $E(X^2) < \infty$ and $E(Y^2) < \infty$. Then,

$$|E(XY)| \leq \sqrt{E(X^2)}\sqrt{E(Y^2)} \tag{1}$$

with equality if and only if $P(X = 0) = 1$ or $P(Y = aX) = 1$ for some $a \in \mathbb{R}$.

1. For $t \in \mathbb{R}$, write $E[(Y - tX)^2]$ as a function of $t$.
2. Using the fact that $E[(Y - tX)^2] \geq 0$ for any $t \in \mathbb{R}$, conclude that either $P(X = 0) = 1$ or

$$E(X^2)\left(t - \frac{E(XY)}{E(X^2)}\right)^2 - \frac{E(XY)^2}{E(X^2)} + E(Y^2) \geq 0$$

   for $t \in \mathbb{R}$.
3. Prove the inequality (1).
4. If $P(X = 0) < 1$, show that equality case in (1) holds if and only if $P(Y = t^*X) = 1$, and give the expression for $t^*$.

   *Hint:* Remember that for $Z$ a non-negative random variable, $E(Z) = 0$ if and only if $P(Z = 0) = 1$.
5. Conclude on the condition for the equality case to hold.