Probability and Statistics

Exercise sheet 13

Exercise 13.1

(a) Let $X_1, ..., X_n$ be i.i.d. $\sim$ Bernoulli(θ_0), for some unknown $\theta_0 \in \Theta = (0, 1)$. Take $\hat{\theta}_n = X_1$ as an estimator of θ_0. We know that $T = T(X_1, ..., X_n) = \sum_{i=1}^n X_i$ is sufficient for this model.

Let $\tilde{\theta}_n = E[\hat{\theta}_n \mid T] = E[X_1 \mid \sum_{i=1}^n X_i]$. Show that $\tilde{\theta}_n = \overline{X}_n$ and compute

$$\text{eff}(\tilde{\theta}_n, \hat{\theta}_n) = \frac{\text{MSE}(\hat{\theta}_n)}{\text{MSE}(\tilde{\theta}_n)}.$$

(b) Let $X_1, ..., X_n$ be i.i.d. $\sim$ U$([0, \theta_0])$, for some $\theta_0 \in \Theta = (0, +\infty)$. Consider $\hat{\theta}_n = 2X_1$ as an estimator of θ_0. Let $T = T(X_1, ..., X_n) = \max_{1 \le i \le n} X_1$. We have shown before that T is sufficient for this model. Let $\tilde{\theta}_n = E[\hat{\theta}_n \mid T] = 2E[X_1 \mid \max_{1 \le i \le n} X_i]$.

Remark: The random vector $(X_1, \max_i X_i)^T$ does not admit a density with respect to the Lebesgue measure on $\mathbb{R}^2$ since $P(X_1 = \max X_i) \neq 0$. Therefore, we will instead compute $\tilde{\theta}_n$ explicitly in the following indirect way.

- You are given the following result: Let $X_1, ..., X_n$ be i.i.d random variables with joint density $\prod_{i=1}^n f(x_i)$ with respect to Lebesgue measure on $(\mathbb{R}^n, \mathcal{B}_{R^n})$. If one orders $X_1, ..., X_n$ in increasing order, we obtain the new random variables $X_{(1)} < ... < X_{(n)}$, the so-called order statistics. For example, $X_{(1)} = \min_{1 \le i \le n} X_i$ and $X_{(n)} = \max_{1 \le i \le n} X_i = T$. The result says that for any $1 \le i < j \le n$, the random vector $(X_{(i)}, X_{(j)})^T$ is absolutely continuous with respect to Lebesgue measure on $(\mathbb{R}^2, \mathcal{B}_{\mathbb{R}^2})$ with density

$$g_{i,j}(s,t) = \frac{n!}{(i-1)!(j-1-i)!(n-j)!} f(s)f(t)[F(s)]^{i-1}[F(t) - F(s)]^{j-1-i}(1 - F(t))^{n-j}\mathbb{1}_{s<t}$$

  where F is the cdf corresponding to f.

  If $j = n$, $i \in \{1, ..., n-1\}$ and the X_i are once again i.i.d. $\sim$ U$([0, \theta_0])$, compute explicitly the joint density $g_{i,n}$ for $(X_{(i)}, X_{(n)})$.
- Find the conditional density of $X_{(i)}$ given $X_{(n)}$, and use it to compute the conditional expectation $E[X_{(i)} \mid X_{(n)}]$.

  Hint: Can you relate this conditional density to a well-known distribution, for which we know the expectation?
- Use a symmetry argument, and the fact that $\sum_{i=1}^n X_i = \sum_{i=1}^n X_{(i)}$, to show that $\tilde{\theta}_n = 2E[X_1 \mid X_{(n)}] = \frac{n+1}{n} X_{(n)}$.
- To conclude, show that $\hat{\theta}_n$ and $\tilde{\theta}_n$ are both unbiased and compute eff$(\tilde{\theta}_n, \hat{\theta}_n)$.

Exercise 13.2 Consider $X_1, ..., X_n$ i.i.d. $\sim$ Exp(λ), $\lambda \in \Theta = (0, +\infty)$. Recall that the pdf of $X_i \sim$ Exp(λ) is given by $f(x \mid \lambda) = \lambda e^{-\lambda x}\mathbb{1}_{x \in (0,+\infty)}$. We want to test $H_0 : \lambda = 1$ versus $H_1 : \lambda = 2$.

(a) Apply the Neyman-Pearson Lemma to find a uniformly most powerful test of level α, based on $X = (X_1, ..., X_n)^T$.

Hint: We recall that if $Y_1, ..., Y_n$ are $\overset{\text{iid}}{\sim}$ Exp(λ_0), then $\sum_{i=1}^n Y_i \sim G(n, \lambda_0)$.

(b) What is the power of the Neyman-Pearson test you found?

Hint: You can express your answer in terms of F_n and F_n^{-1}, the cdf and inverse cdf of a $\mathrm{G}(n,1)$ distribution.

(c) For $n = 10$, we observe the following sample:

1.009	0.132	0.384	0.360	0.206	0.588	0.872	0.398	0.339	1.079

What decision do you take, if you want the level of the test to be equal to $\alpha = 0.05$? What about $\alpha = 0.01$?

Hint: The quantiles of the $G(10, 1)$ distribution of order 5% and 1% are 5.425 and 4.130, respectively.

Exercise 13.3 Again in the setup of exercise 2, it turns out that the Neyman-Pearson test you found in (a) is actually UMP of level α for testing $H_0 : \lambda = 1$ versus $H_1' : \lambda > 1$. More concretely, the same NP test is the most powerful among all tests of level α, for any $\lambda \in \Theta_1' = (1, +\infty)$, and not only for $\lambda \in \Theta_1 = \{2\}$.

Do you see why this is true?