Probability and Statistics

Exercise sheet 2

Exercise 2.1 Birthday problem (without Stirling's approximation).

Consider again n people assumed to have the same probability of being born on any day of the year calendar. Assume also that this calendar consists of 365 days. Write

$$A_n = \{\text{at least 2 people in the group have the same birthday}\}.$$

The aim of this exercise is to find bounds for $P(A_n^c)$ for any fixed $2 \leq n \leq 365 = N$.

(a) Show that for any two given people i and j ($1 \leq i \neq j \leq n$) from the group,

$$P(i \text{ and } j \text{ have the same birthday}) = \frac{1}{N}.$$

(b) Using (a), and the known inequality

$$\log(1+x) \leq x, \quad x \in (-1, \infty)$$

show that

$$1 - \frac{n(n-1)}{2N} \leq P(A_n^c) \leq \exp\left(-\frac{n(n-1)}{2N}\right).$$

(c) Find that $n_m = 23$ is the smallest n so that $P(A_n) \geq \frac{1}{2}$.

Exercise 2.2 We have a box which contains 3 different coins. Each of these coins has a different probability to show heads after it is tossed. Call these probabilities $p_j, j = 1, 2, 3$. We know that

$$p_1 = \frac{1}{4}, \quad p_2 = \frac{1}{2}, \quad p_3 = \frac{3}{4}.$$

(a) We select a coin from the box completely at random. When this coin is tossed, it shows heads. What is the conditional probability that coin j was selected?

(b) The same coin is tossed again. What is the conditional probability of obtaining heads again?

(c) [1] Prove the following result:
Let $A_1, ..., A_k$ be a partition of Ω, and B, C events in Ω with $P(B \cap C) > 0$ and $P(A_i \cap B) > 0$ for $i = 1, ..., k$. Then,

$$P(A_j \mid B \cap C) = \frac{P(A_j \mid B) P(C \mid A_j \cap B)}{\sum_{i=1}^{k} P(A_i \mid B) P(C \mid A_i \cap B)}.$$

(This result is called the conditional Bayes' theorem.)

(d) If the same coin shows again heads at the second toss, what is the conditional probability that coin j was selected?

[1] This part was updated a posteriori.

Exercise 2.3 (Simpson's paradox).

We are interested in studying the probability of success of a student at an entrance exam for two departments of a university. Consider the following events:

$$A := \{\text{The student is a man}\}$$
$$A^c = \{\text{The student is a woman}\}$$
$$B := \{\text{The student applied for department I}\}$$
$$B^c = \{\text{The student applied for department II}\}$$
$$C := \{\text{The student was accepted}\}$$
$$C^c = \{\text{The student was not accepted}\}$$

We assume the following probabilities:

$$P(A) = 0.73,$$
$$P(B \mid A) = 0.69, P(B \mid A^c) = 0.24,$$
$$P(C \mid A \cap B) = 0.62, P(C \mid A^c \cap B) = 0.82,$$
$$P(C \mid A \cap B^c) = 0.06, P(C \mid A^c \cap B^c) = 0.07.$$

(a) Draw a tree describing the situation with the probabilities associated.

(b) From examining the probabilities in the tree, do you think that in the selection process the women are disadvantaged?

(c) Calculate $P(C \mid A)$ and $P(C \mid A^c)$. Do you agree with your answer in 3.2? What is going on?