Probability and Statistics

Exercise sheet 3

Exercise 3.1 In a clinical trial with two treatment groups, the probability of success (the patient being cured) in one treatment group is $p_1 = 0.5$, and the probability of success in the other is $p_2 = 0.6$. There are 5 patients in each group. Assuming the outcomes for all patients are independent, calculate the probability that the first treatment group has at least as many successes as in the second one.

Exercise 3.2 Suppose a fair die is rolled once and the observed number is $N \in \{1, ..., 6\}$. Then, a fair coin is tossed N times. Let X be the number of heads obtained.

Find the pmf, cdf and expectation of X. Does the expected value make sense intuitively?

Exercise 3.3

(a) Suppose that X has pmf

$$P\left(X = \frac{1}{n}\right) = \frac{1}{2^n} \quad (n \geq 1).$$

Find $E(X)$.

(b) Suppose that X has pmf

$$P\left(X = \frac{1}{n}\right) = \frac{1}{2^{n+1}} \quad (n \geq 1)$$

and

$$P(X = n) = \frac{1}{2^n} \quad (n \geq 2).$$

Find $E(X)$.

Exercise 3.4

(a) Fix p a positive integer. Give an example of a random variable X taking values in $\{1, 2, 3, ...\}$ such that $E(X^k) < \infty \ \forall k < p$ but $E(X^p) = \infty$.

(b) Let X be some nonnegative random variable and p some positive integer. Show that $E(X^p) \geq E(X)^p$. Can we have equality?

Exercise 3.5 (optional) Let $s \in (1, \infty)$. The Riemann zeta function is defined as

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

The goal is to prove that

$$\zeta(s) = \frac{1}{\prod_{i=1}^{\infty} \left(1 - \frac{1}{p_i^s}\right)}$$

with $p_1 = 2$, $p_2 = 3$, $p_3 = 5$, ... the prime numbers (in order).

(a) Take $(\mathbb{N}, 2^{\mathbb{N}}, P)$ with
$$P(A) = \frac{1}{\zeta(s)} \sum_{n \in A} \frac{1}{n^s}$$
for any $A \in 2^{\mathbb{N}}$.
Show that $(\mathbb{N}, 2^{\mathbb{N}}, P)$ is a probability space.

(b) For p a prime number, let
$$N_p = \{n \in \mathbb{N} : n \text{ divisible by } p\}.$$
Calculate $P(N_p)$.

(c) Prove that events $\{N_{p_i}\}_{i \geq 1}$ are mutually independent.

(d) Compute $P(\cap_{i \geq 1} N_{p_i}^c)$ and conclude.