Probability and Statistics

Exercise sheet 5

Exercise 5.1 Let X be a real-valued random variable defined on a probability space $(\Omega, \mathcal{A}, P)$. For a fixed integer $k \in \{1, 2, ...\}$ show that $E(X^k)$ exists if and only if $E[(X - E(X))^k]$ exists. In other words, you need to show that

$$E(|X|^k) < \infty \;\Leftrightarrow\; E[|X - E(X)|^k] < \infty.$$

(The case $k = 1$ is trivially true).

Exercise 5.2 (Proving Jensen's inequality).

Let φ be a convex function defined on an interval (a, b) with $-\infty \leq a < b \leq +\infty$. Consider some random variable X such that $P(X \in (a, b)) = 1$. Assume that $E(X)$ and $E(\varphi(X))$ exist, that is, $E(|X|) < \infty$ and $E(|\varphi(X)|) < \infty$. Here, we recall that φ is convex on (a, b) if $\forall x, y \in (a, b)$ and $\lambda \in [0, 1]$

$$\varphi(\lambda x + (1 - \lambda)y) \leq \lambda\varphi(x) + (1 - \lambda)\varphi(y)$$

and that it is strictly convex if $\forall x, y \in (a, b)$ such that $x \neq y$ and $\lambda \in (0, 1)$

$$\varphi(\lambda x + (1 - \lambda)y) < \lambda\varphi(x) + (1 - \lambda)\varphi(y).$$

(a) Show that for any $c \in (a, b)$, we can find a linear function l such that

$$\varphi(x) \geq l(x) \quad \forall x \in (a, b)$$

and

$$\varphi(c) = l(c).$$

Hint: You may assume that φ admits left and right derivatives, i.e. the limits

$$\varphi_+(x) := \lim_{\epsilon \to 0^+} \frac{\varphi(x + \epsilon) - \varphi(x)}{\epsilon}$$

and

$$\varphi_-(x) := \lim_{\epsilon \to 0^-} \frac{\varphi(x + \epsilon) - \varphi(x)}{\epsilon}$$

both exist for any $x \in (a, b)$. Explain why $\varphi_+(x) \geq \varphi_-(x)$ for $x \in (a, b)$. Then, show that for any $s \in [\varphi_-(c), \varphi_+(c)]$ you can construct a line with slope s with the desired properties.

(b) Show that under the given assumptions on X we have

$$E[\varphi(X)] \geq \varphi(E(X)).$$

Hint: Use your result from (a). Given the random variable X, what is a reasonable value to choose for c?

(c) Suppose now that φ is strictly convex. Show that we have equality if and only if $P(X = E(X)) = 1$, that is, X is a degenerate random variable.

Exercise 5.3 Suppose you can choose a number $n \geq 1$ and then toss a fair coin n times. You will be given a prize if you get either exactly 7 or exactly 9 heads. What is the "best" choice for the number n?

Exercise 5.4 (A novel way to give a test)

A student takes a 5-answer multiple choice test. His/her grade is determined by the number of questions required to get 5 correct answers. The grading is done as follows:
- Grade A is given if the student only needs 5 questions;
- Grade B is given if the student needs 6 or 7 questions;
- Grade C is given if the student needs 8 or 9 questions;
- Grade F (fail) is given otherwise.

Suppose the student guesses independently at random on each question. What is the most likely grade (i.e. which outcome has the highest probability)?

Exercise 5.5 (Generating functions). (Optional)

Let X be some integer-valued random variable, that is $X(\omega) \in \{0, 1, 2, ...\}$ $\forall \omega \in \Omega$, the sample space on which X is defined. The generating function of X is defined as

$$G(s) := \sum_{k=0}^{\infty} s^k P(X = k),$$

for those values of s such that the sum on the right-hand side converges.
Note that G is always well-defined for $|s| \leq 1$ since

$$\sum_{k=0}^{\infty} |s|^k P(X = k) \leq \sum_{k=0}^{\infty} P(X = k) = 1.$$

Also, $G(s) = E[s^X]$ is another expression for G.

(a) Consider a power series $f(s) = \sum_{k=0}^{\infty} a_k s^k$, given a real sequence $(a_k)_{k \geq 0}$ and $s \in \mathbb{R}$ for which $f(s)$ is defined. Suppose that there is some $s_0 \neq 0$ such that $f(s_0)$ is defined, that is $\sum_{k=0}^{\infty} a_k s_0^k$ converges.

Show that f is defined and infinitely differentiable for all s such that $|s| < |s_0|$ and

$$f^{(j)}(s) = \sum_{k=j}^{\infty} a_k k(k-1)...(k-j+1) s^{k-j}$$

for $|s| < |s_0|$.

Hint: You may use the fact that if $(f_n)_{n=1}^{\infty}$ is a sequence of differentiable functions $f_n : (a, b) \to \mathbb{R}$ such that $f_n \to f$ pointwise on (a, b) and $f_n' \to g$ uniformly on (a, b) for some functions $f, g : (a, b) \to \mathbb{R}$, then $f' = g$. You may need to apply this inductively to conclude.

(b) Conclude from (a) that $\forall s : |s| < 1$, the generating function G defined above is infinitely differentiable and compute $G^{(j)}(0)$.

(c) Let $X \sim \text{Unif}\{1, 2, ..., n\}$ for some $n \geq 1$. What is the expression of G_X, the generating function of such an X?

(d) Consider now two random variables X and Y that are i.i.d $\sim \text{Unif}\{1, 2, ..., n\}$. Let $S = X + Y$. What is the expression of G_S, the generating function of S?

(e) Use (b) to find the pmf of S. Is it easier to do it directly?