Probability and Statistics

Exercise sheet 8

Exercise 8.1

(a) Let $X \sim \mathrm{U}([0,1])$. Compute $E(X^n)$, $E(X^{\frac{1}{n}})$ $(n \geq 1)$, and $\Psi_X(t) = E[e^{tX}]$ whenever it is defined.

(b) Let $X \sim \mathrm{Beta}(\alpha, \beta)$, $\alpha > 0$ and $\beta > 0$. Compute $E(X)$ and $\mathrm{var}(X)$.

Hint: Use the "trick" that any density function f has to integrate to 1.

(c) Let $X \sim \mathrm{Exp}(\lambda)$, for $\lambda > 0$. Compute the cdf of X and $E(X^n)$ for $n \geq 1$.

Remark: Watch out for the parametrisation - different sources may use different parametrisations. In the lectures we consider the density function of an $\mathrm{Exp}(\lambda)$-distributed random variable to be $f(x) = \lambda e^{-\lambda x}$ for $x \geq 0$.

Exercise 8.2 (Waiting time.)
An auto towing company services a 50 mile stretch of a highway. The company is located 20 miles from one end of the stretch. Breakdowns occur uniformly along the highway and the towing trucks travel at a constant speed of 50mph. Find the mean and variance of the time elapsed between the instant the company is called and a towing truck arrives.

Exercise 8.3 (Uniforms, uniforms...)

(a) Consider a random variable $X \sim \mathrm{U}\left(\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]\right)$. Find $E(\sin(X))$ and $\mathrm{var}(\sin(X))$.

(b) The lengths of the sides of a triangle are X, $2X$ and $2.5X$ with $X \sim \mathrm{U}([0, \alpha])$ for some $\alpha > 0$.

- Find the mean and variance of its area.
  Hint: Recall that if
  $$s = \frac{a+b+c}{2}$$
  with a, b, c the lengths, then the area of the triangle is
  $$|\Delta| = \sqrt{s(s-a)(s-b)(s-c)}$$
  (Heron's formula).
- How should we choose α so that the mean area is ≥ 1?

(c) Take $X_1, ..., X_n$ to be $\stackrel{\text{iid}}{\sim} \mathrm{U}([0,1])$. Let $M_n = \max(X_1, ..., X_n)$. Find the cdf and pdf of M_n. Can you recognise this distribution? What are $E(M_n)$ and $\mathrm{var}(M_n)$?

Exercise 8.4 (Quantile transformation.) Recall that for a given cdf F, the quantile t_α of order $\alpha \in (0,1)$ is defined as

$$t_\alpha = \inf\{t : F(t) \geq \alpha\} =: F^{-1}(\alpha).$$

F^{-1} denotes the generalised inverse of F. When the latter is bijective (at least in the neighbourhood of t_α), then F^{-1} is the inverse of F in the classical sense.

(a) Consider $U \sim \mathrm{U}([0,1])$. Show that $1-U \sim \mathrm{U}([0,1])$.
Hint: Compute, for example, the cdf or the pdf of $1-U$.

(b) Consider
$$X := -\frac{1}{\lambda}\log(U)$$
(for $\omega : U(\omega) = 0$, take $X(\omega) = 0$, say).
Find the cdf of X. Can you recognise this distribution?

(c) Now, consider the following problem: take a cdf F which is bijective when viewed as a map $F : (a,b) \to (0,1)$, for some $-\infty \le a < b \le +\infty$.
Define $X = F^{-1}(U)$, with $U \sim \mathrm{U}([0,1])$.

- Compute the cdf of X.
  Hint: You may need to consider the cases $a = -\infty$, $a > -\infty$, $b = +\infty$, $b < +\infty$.
- Compute the pdf of X, assuming that F is C^1 on (a,b) with $F'(x) > 0\ \forall x \in (a,b)$.
- Can you make the link to (b)?

(d) Suppose you are given a numerical algorithm which enables you to generate a random number from $[0,1]$. You would like to generate a random number X which follows the Cauchy distribution, i.e.
$$f_X(x) = \frac{1}{\pi(1+x^2)},\ x \in \mathbb{R}.$$
Can you propose a way to do that, based on your previous findings?

Exercise 8.5 (Optional.) Let $X_1, ..., X_n$ (for some $n \ge 2$) be random variables defined on the same probability space $(\Omega, \mathcal{A}, P)$. A necessary and sufficient condition for (mutual) independence of $X_1, ..., X_n$ is that
$$P(X_1 \le x_1, ..., X_n \le x_n) = \prod_{i=1}^{n} P(X_i \le x_i) \tag{1}$$
for any $(x_1, ..., x_n) \in \mathbb{R}^n$.

The goal of this exercise is to show that when $X_1, ..., X_n$ are discrete, (1) is equivalent to
$$P(X_{i_1} = x_1, ..., X_{i_m} = x_m) = \prod_{j=1}^{m} P(X_{i_j} = x_j) \tag{2}$$
for any $1 \le m \le n$ and $i_1 < ... < i_m$, $(x_1, ..., x_m) \in \mathbb{R}^m$.

(a) Take $m = n$ (in which case $i_1 = 1, i_2 = 2, ..., i_m = n$).

- Focus on X_1 and show that (2) implies that for any $(x_1, ..., x_n) \in \mathbb{R}^n$,
$$P(X_1 \le x_1, X_2 = x_2, ..., X_n = x_n) = P(X_1 \le x_1)P(X_2 = x_2)...P(X_n = x_n).$$
- Repeating this argument inductively, show that (1) holds.

(b) Now, we want to show that (1) implies (2). Fix $\{i_1, ..., i_m\} = \mathcal{J} \subset \{1, ..., n\}$, with strict inclusion.

- Show that (1) implies that

$$P(X_{i_1} \leq x_1, ..., X_{i_m} \leq x_m) = \prod_{j=1}^{m} P(X_{i_j} \leq x_j).$$

Hint: You may want to take limits in (1) as $x_i \to +\infty$ for $i \notin \mathcal{J}$.

- Focus on i_1 (hold the other events depending on $i_2, ..., i_m$ fixed).
Show that we have

$$P(X_{i_1} < x_1, X_{i_2} \leq x_2, ..., X_{i_m} \leq x_m) = P(X_{i_1} < x_1)P(X_{i_2} \leq x_2)...P(X_{i_m} \leq x_m)$$

for any $(x_1, ..., x_m) \in \mathbb{R}^m$.
Hint: Consider $A_k := \{X_{i_1} \leq x_1 - \frac{1}{k}, X_{i_2} \leq x_2, ..., X_{i_m} \leq x_m\}$ and use the monotone convergence theorem.

- Conclude that

$$P(X_{i_1} = x_1, X_{i_2} \leq x_2, ..., X_{i_m} \leq x_m) = P(X_{i_1} = x_1)P(X_{i_2} \leq x_2)...P(X_{i_m} \leq x_m).$$

- Repeat your argument inductively to conclude.