Probability and Statistics

Exercise sheet 13

Exercise 13.1

(a) Let $X_1, ..., X_n$ be i.i.d. $\sim$ Bernoulli(θ_0), for some unknown $\theta_0 \in \Theta = (0, 1)$. Take $\hat{\theta}_n = X_1$ as an estimator of θ_0. We know that $T = T(X_1, ..., X_n) = \sum_{i=1}^n X_i$ is sufficient for this model.

Let $\tilde{\theta}_n = E[\hat{\theta}_n \mid T] = E[X_1 \mid \sum_{i=1}^n X_i]$. Show that $\tilde{\theta}_n = \overline{X}_n$ and compute

$$\text{eff}(\tilde{\theta}_n, \hat{\theta}_n) = \frac{\text{MSE}(\hat{\theta}_n)}{\text{MSE}(\tilde{\theta}_n)}.$$

(b) Let $X_1, ..., X_n$ be i.i.d. $\sim$ U($[0, \theta_0]$), for some $\theta_0 \in \Theta = (0, +\infty)$. Consider $\hat{\theta}_n = 2X_1$ as an estimator of θ_0. Let $T = T(X_1, ..., X_n) = \max_{1 \le i \le n} X_1$. We have shown before that T is sufficient for this model. Let $\tilde{\theta}_n = E[\hat{\theta}_n \mid T] = 2E[X_1 \mid \max_{1 \le i \le n} X_i]$.

Remark: The random vector $(X_1, \max_i X_i)^T$ does not admit a density with respect to the Lebesgue measure on $\mathbb{R}^2$ since $P(X_1 = \max X_i) \neq 0$. Therefore, we will instead compute $\tilde{\theta}_n$ explicitly in the following indirect way.

- You are given the following result: Let $X_1, ..., X_n$ be i.i.d random variables with joint density $\prod_{i=1}^n f(x_i)$ with respect to Lebesgue measure on $(\mathbb{R}^n, \mathcal{B}_{R^n})$. If one orders $X_1, ..., X_n$ in increasing order, we obtain the new random variables $X_{(1)} < ... < X_{(n)}$, the so-called order statistics. For example, $X_{(1)} = \min_{1 \le i \le n} X_i$ and $X_{(n)} = \max_{1 \le i \le n} X_i = T$. The result says that for any $1 \le i < j \le n$, the random vector $(X_{(i)}, X_{(j)})^T$ is absolutely continuous with respect to Lebesgue measure on $(\mathbb{R}^2, \mathcal{B}_{\mathbb{R}^2})$ with density

  $$g_{i,j}(s, t) = \frac{n!}{(i-1)!(j-1-i)!(n-j)!} f(s) f(t) [F(s)]^{i-1} [F(t) - F(s)]^{j-1-i} (1 - F(t))^{n-j} \mathbb{1}_{s<t}$$

  where F is the cdf corresponding to f.

  If $j = n$, $i \in \{1, ..., n-1\}$ and the X_i are once again i.i.d. $\sim$ U($[0, \theta_0]$), compute explicitly the joint density $g_{i,n}$ for $(X_{(i)}, X_{(n)})$.
- Find the conditional density of $X_{(i)}$ given $X_{(n)}$, and use it to compute the conditional expectation $E[X_{(i)} \mid X_{(n)}]$.

  Hint: Can you relate this conditional density to a well-known distribution, for which we know the expectation?
- Use a symmetry argument, and the fact that $\sum_{i=1}^n X_i = \sum_{i=1}^n X_{(i)}$, to show that $\tilde{\theta}_n = 2E[X_1 \mid X_{(n)}] = \frac{n+1}{n} X_{(n)}$.
- To conclude, show that $\hat{\theta}_n$ and $\tilde{\theta}_n$ are both unbiased and compute $\text{eff}(\tilde{\theta}_n, \hat{\theta}_n)$.

Solution 13.1

(a) By symmetry of the distribution, one can see that the conditional expectation must satisfy

$$E\left[X_1 \mid \sum_{i=1}^n X_i\right] = E\left[X_2 \mid \sum_{i=1}^n X_i\right] = ... = E\left[X_n \mid \sum_{i=1}^n X_i\right]$$

and so

$$E\left[X_1 \mid \sum_{i=1}^n X_i\right] = \frac{1}{n} E\left[\sum_{i=1}^n X_i \mid \sum_{i=1}^n X_i\right] = \frac{1}{n}\sum_{i=1}^n X_i.$$

Alternatively, we also show how to compute this directly. If $n = 1$, clearly $E[X_1 \mid X_1] = X_1 = \overline{X}_1$, so we may assume $n \geq 2$.

First we find the joint pmf of $(X_1, \sum_{i=1}^n X_i)$. If p is the joint pmf,

$$\begin{aligned} p(x,t) &= P\left(X_1 = x, \sum_{i=1}^n X_i = t\right) \\ &= P\left(X_1 = x, \sum_{i=2}^n X_i = t - x\right) \\ &= P(X_1 = x) P\left(\sum_{i=2}^n X_i = t - x\right) \end{aligned}$$

by independence. (More specifically, independence of $X_1, ..., X_n$ implies that X_1 and $f(X_2, ..., X_n)$ are independent, for $f(x_2, ..., x_n) = \sum_{i=2}^n x_i$.)

Now, since $X_2, ..., X_n$ are i.i.d $\sim$ Bernoulli(θ_0), it follows that $\sum_{i=2}^n X_i \sim \text{Bin}(n-1, \theta_0)$. Hence,

$$\begin{aligned} p(x,t) &= \begin{cases} \theta_0^x (1-\theta_0)^{1-x} \binom{n-1}{t-x} \theta_0^{t-x} (1-\theta_0)^{n-1-t+x}, & \text{if } x \in \{0,1\}, t - x \in \{0, ..., n-1\} \\ 0, & \text{otherwise} \end{cases} \\ &= \begin{cases} \binom{n-1}{t-x} \theta_0^t (1-\theta_0)^{n-t}, & \text{if } x \in \{0,1\}, t \in \{x, ..., x+n-1\} \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

The conditional pmf of X_1 given $\sum_{i=1}^n X_i = t$ is given by

$$p(x \mid t) = \frac{p(x,t)}{p_T(t)}$$

wherever $p_T(t) > 0$. Note that $T = \sum_{i=1}^n X_i$ takes values in $\{0, 1, ..., n\}$, so we can consider only t in that set. Moreover, T has a Bin(n, θ_0) distribution, so that

$$p_T(t) = \binom{n}{t} \theta_0^t (1-\theta_0)^{n-t}, \quad t \in \{0, ..., n\}.$$

Hence

$$p(x \mid t) = \begin{cases} \frac{\binom{n-1}{t-x}}{\binom{n}{t}}, & \text{if } x \in \{0,1\}, t \in \{x, ..., x+n-1\} \\ 0, & \text{otherwise.} \end{cases}$$

(note that $\{x, ..., x+n-1\} \subset \{0, ..., n\}$ for $x \in \{0,1\}$).

We can now compute for $t \in \{0, ..., n\}$:

$$\begin{aligned} g(t) = E\left[X_1 \mid \sum_{i=1}^n X_i = t\right] &= \sum_{x\in\{0,1\}} xp(x \mid t) \\ &= p(1 \mid t) \\ &= \frac{\binom{n-1}{t-1}}{\binom{n}{t}} \mathbb{1}_{t\in\{1,\ldots,n\}} \\ &= \frac{t}{n}\mathbb{1}_{t\in\{1,\ldots,n\}}. \end{aligned}$$

Noticing that $\frac{t}{n} = 0$ in the case that $t = 0$, we can write more simply that

$$g(t) = E\left[X_1 \mid \sum_{i=1}^n X_i = t\right] = \frac{t}{n}$$

for $t \in \{0, 1, ..., n\}$.

Thus,

$$E\left[X_1 \mid \sum_{i=1}^n X_i\right] = \frac{1}{n}\sum_{i=1}^n X_i = \overline{X}_n$$

with probability 1, since $\sum_{i=1}^n X_i \in \{0, 1, ..., n\}$ with probability 1.

Therefore, by either method, we conclude that $\tilde{\theta}_n = \overline{X}_n$ (with probability 1).

Now note that

$$\begin{aligned} E[\hat{\theta}_n] &= E[X_1] = \theta_0, \\ E[\tilde{\theta}_n] &= \frac{1}{n}\sum_{i=1}^n \theta_0 = \theta_0 \end{aligned}$$

so that $\hat{\theta}_n$, $\tilde{\theta}_n$ are both unbiased. Therefore,

$$\mathrm{MSE}(\hat{\theta}_n) = \mathrm{var}(\hat{\theta}_n) + \mathrm{bias}(\hat{\theta}_n)^2 = \mathrm{var}(X_1) = \theta_0(1-\theta_0)$$

and

$$\mathrm{MSE}(\tilde{\theta}_n) = \mathrm{var}(\tilde{\theta}_n) + \mathrm{bias}(\tilde{\theta}_n)^2 = \mathrm{var}(\overline{X}_n) = \frac{1}{n}\theta_0(1-\theta_0).$$

Finally,

$$\mathrm{eff}(\tilde{\theta}_n, \hat{\theta}_n) = \frac{\mathrm{var}(\hat{\theta}_n)}{\mathrm{var}(\tilde{\theta}_n)} = \frac{\theta_0(1-\theta_0)}{\frac{\theta_0(1-\theta_0)}{n}} = n.$$

(b)
- Using the result, we find the joint density of $(X_{(i)}, X_{(n)})$ to be:

$$\begin{aligned} g_{i,n}(s,t) &= \frac{n!}{(i-1)!(n-1-i)!(n-n)!}\frac{\mathbb{1}_{0\le s\le\theta_0}\mathbb{1}_{0\le t\le\theta_0}}{\theta_0^2}\left(\frac{s}{\theta_0}\right)^{i-1}\left(\frac{t-s}{\theta_0}\right)^{n-1-i}\left(\frac{1-t}{\theta_0}\right)^{n-n}\mathbb{1}_{s<t} \\ &= \frac{n!}{(i-1)!(n-1-i)!}\frac{1}{\theta_0^n}s^{i-1}(t-s)^{n-1-i}\mathbb{1}_{0\le s<t\le\theta_0}. \end{aligned}$$

- First we find the marginal density of $X_{(n)}$. We have, for $t \in [0, \theta_0]$:

$$\begin{aligned}
f_{X_{(n)}}(t) &= \int g_{i,n}(s,t)ds \\
&= \int \frac{n!}{(i-1)!(n-1-i)!} \frac{1}{\theta_0^n} s^{i-1}(t-s)^{n-1-i} \mathbb{1}_{0 \le s < t \le \theta_0} ds \\
&= \int_0^t \frac{n!}{(i-1)!(n-1-i)!} \frac{1}{\theta_0^n} s^{i-1}(t-s)^{n-1-i} ds \\
&\overset{ts'=s}{=} \frac{n!}{(i-1)!(n-1-i)!} \frac{1}{\theta_0^n} \int_0^1 (ts')^{i-1}(t-ts')^{n-1-i} \frac{ds'}{t} \\
&= \frac{n!}{(i-1)!(n-1-i)!} \frac{t^{n-1}}{\theta_0^n} \int_0^1 (s')^{i-1}(1-s')^{n-1-i} ds' \\
&= \frac{n!}{(i-1)!(n-1-i)!} \frac{t^{n-1}}{\theta_0^n} \frac{\Gamma(i)\Gamma(n-i)}{\Gamma(n)} \\
&= \frac{n!(i-1)!(n-i-1)!}{(n-1)!(i-1)!(n-1-i)!} \frac{t^{n-1}}{\theta_0^n} \\
&= n\frac{t^{n-1}}{\theta_0^n}.
\end{aligned}$$

Clearly, $f_{X_{(n)}}(t) = 0$ outside of $[0, \theta_0]$. We can note that $X_{(n)}$ has a Beta$(n, 1)$ distribution, rescaled to the interval $[0, \theta_0]$ rather than $[0, 1]$. In other words, $\frac{X_{(n)}}{\theta_0} \sim$ Beta$(n, 1)$, as we can check using the Jacobian formula.
Alternatively, one could find the marginal cdf of $X_{(n)}$ by

$$\begin{aligned}
F_{X_{(n)}}(t) &= P(\max_{1 \le i \le n} X_i \le t) \\
&= P(X_1 \le t, ..., X_n \le t) \\
&= P(X_1 \le t)...P(X_n \le t) \\
&= \begin{cases} 0, & t < 0 \\ \frac{t^n}{\theta_0^n}, & 0 \le t < \theta_0 \\ 1, & \theta_0 \le t. \end{cases}
\end{aligned}$$

Since this is piecewise C^1, we can differentiate to find the same marginal density as above.
Now we have what we want for finding the conditional distribution. Assume that $t \in (0, \theta_0)$, since otherwise the marginal density of $X_{(n)}$ vanishes. Then:

$$\begin{aligned}
f_i(s \mid t) &= \frac{g_{i,n}(s,t)}{f_{X_{(n)}}(t)} \\
&= \frac{\frac{n!}{(i-1)!(n-1-i)!} \frac{1}{\theta_0^n} s^{i-1}(t-s)^{n-1-i} \mathbb{1}_{0 \le s < t}}{n\frac{t^{n-1}}{\theta_0^n}} \\
&= \frac{(n-1)!}{(i-1)!(n-1-i)!} \frac{1}{t^{n-1}} s^{i-1}(t-s)^{n-1-i} \mathbb{1}_{0 \le s < t}
\end{aligned}$$

Then for any fixed $t \in [0, \theta_0]$, we can see that we get a rescaled Beta distribution as the conditional distribution for X_i. We can write it as follows: for $Y \sim$ Beta$(i, n-i)$,

$$X_{(i)} \mid (X_{(n)} = t) \overset{\mathrm{d}}{=} tY.$$

This can again be checked by the Jacobian formula, using $h(u) = ut$.
With this representation it is now easy to find the conditional expectation:

$$E[X_{(i)} \mid X_{(n)} = t] \ = E[tY] = t\frac{i}{n},$$

or in other words,

$$E[X_{(i)} \mid X_{(n)}] = \frac{i}{n} X_{(n)}.$$

- Note that, since $X_{(n)} = \max(X_1, ..., X_n)$ is symmetric with respect to $X_1, ..., X_n$, and since the joint distribution of the X_i is symmetric (since they are i.i.d), we obtain that

$$E[X_1 \mid X_{(n)}] = E[X_2 \mid X_{(n)}] = ...E[X_n \mid X_{(n)}].$$

Therefore, we also get

$$E[X_1 \mid X_{(n)}] = \frac{1}{n} E\left[\sum_{i=1}^{n} X_i \mid X_{(n)}\right].$$

Now, $\sum_{i=1}^{n} X_i = \sum_{i=1}^{n} X_{(i)}$ since both sums contain the same terms, just in potentially different orders. Therefore,

$$E[X_1 \mid X_{(n)}] = \frac{1}{n} E\left[\sum_{i=1}^{n} X_{(i)} \mid X_{(n)}\right].$$

We can compute this term on the right hand side, which gives us our answer:

$$\begin{aligned}
E[X_1 \mid X_{(n)}] &= \frac{1}{n} E\left[\sum_{i=1}^{n} X_{(i)} \mid X_{(n)}\right] \\
&= \frac{1}{n} \sum_{i=1}^{n} \frac{i}{n} X_{(n)} \\
&= \frac{1}{n^2} X_{(n)} \sum_{i=1}^{n} i \\
&= \frac{1}{n^2} X_{(n)} \frac{n(n+1)}{2} \\
&= \frac{n+1}{2n} X_{(n)}.
\end{aligned}$$

as we wanted, i.e. $2E[X_1 \mid X_{(n)}] = \frac{n+1}{n} X_{(n)}$.

- $$E[\hat{\theta}_n] = E[2X_1] = 2\frac{\theta_0}{2} = \theta_0$$

and

$$E[\tilde{\theta}_n] = E\left[\frac{n+1}{n} X_{(n)}\right] = \frac{n+1}{n} \theta_0 \frac{n}{n+1} \ = \theta_0,$$

since, as we noted above, $X_{(n)}$ has a rescaled Beta distribution ($X_{(n)} \stackrel{d}{=} \theta_0 Z$ for $Z \sim \text{Beta}(n, 1)$). Alternatively, we could obtain the same result by using the law of iterated expectation. Either way, we see that $\hat{\theta}_n$ and $\tilde{\theta}_n$ are unbiased.

We can then compute the variances, using the variance of the uniform and Beta distributions, respectively:

$$\text{var}(\hat{\theta}_n) = \text{var}(2X_1) = 4\frac{\theta^2}{12} = \frac{\theta^2}{3}$$

and

$$\text{var}(\tilde{\theta}_n) = \text{var}\left(\frac{n+1}{n}X_{(n)}\right) = \frac{(n+1)^2\theta_0^2}{n^2}\frac{n}{(n+1)^2(n+2)} = \frac{\theta_0^2}{n(n+2)}.$$

Finally,

$$\begin{aligned}
\text{eff}(\tilde{\theta}_n, \hat{\theta}_n) &= \frac{\text{MSE}(\hat{\theta}_n)}{\text{MSE}(\tilde{\theta}_n)} \\
&= \frac{\text{var}(\hat{\theta}_n)}{\text{var}(\tilde{\theta}_n)} \quad (\text{as } \hat{\theta}_n, \tilde{\theta}_n \text{ are unbiased}) \\
&= \frac{\theta_0^2}{3}\frac{n(n+2)}{\theta_0^2} \\
&= \frac{n(n+2)}{3}.
\end{aligned}$$

In this case we get that $\tilde{\theta}_n$ is a much more efficient estimator.

Exercise 13.2 Consider $X_1, ..., X_n$ i.i.d. $\sim \text{Exp}(\lambda)$, $\lambda \in \Theta = (0, +\infty)$. Recall that the pdf of $X_i \sim \text{Exp}(\lambda)$ is given by $f(x \mid \lambda) = \lambda e^{-\lambda x}\mathbb{1}_{x\in(0,+\infty)}$. We want to test $H_0 : \lambda = 1$ versus $H_1 : \lambda = 2$.

(a) Apply the Neyman-Pearson Lemma to find a uniformly most powerful test of level α, based on $X = (X_1, ..., X_n)^T$.

Hint: We recall that if $Y_1, ..., Y_n$ are $\overset{\text{iid}}{\sim} \text{Exp}(\lambda_0)$, then $\sum_{i=1}^n Y_i \sim G(n, \lambda_0)$.

(b) What is the power of the Neyman-Pearson test you found?

Hint: You can express your answer in terms of F_n and F_n^{-1}, the cdf and inverse cdf of a $\text{G}(n, 1)$ distribution.

(c) For $n = 10$, we observe the following sample:

1.009	0.132	0.384	0.360	0.206	0.588	0.872	0.398	0.339	1.079

What decision do you take, if you want the level of the test to be equal to $\alpha = 0.05$? What about $\alpha = 0.01$?

Hint: The quantiles of the $G(10, 1)$ distribution of order 5% and 1% are 5.425 and 4.130, respectively.

Solution 13.2

(a) The NP-test is given in the form

$$d_{NP}(\mathbf{x}) = \begin{cases} 1, & \frac{f_{\mathbf{x}}(\mathbf{x}|\lambda_1)}{f_{\mathbf{x}}(\mathbf{x}|\lambda_0)} > k_\alpha \\ \gamma_\alpha, & \frac{f_{\mathbf{x}}(\mathbf{x}|\lambda_1)}{f_{\mathbf{x}}(\mathbf{x}|\lambda_0)} = k_\alpha \\ 0, & \frac{f_{\mathbf{x}}(\mathbf{x}|\lambda_1)}{f_{\mathbf{x}}(\mathbf{x}|\lambda_0)} < k_\alpha, \end{cases}$$

for some suitable $k_\alpha > 0$ and $\gamma_\alpha \in [0, 1]$, such that $E_{\lambda_0}[d_{NP}(\mathbf{X})] = \alpha$. A value of 1 corresponds to rejecting the null hypothesis, and a value of 0 corresponds to not rejecting the null hypothesis. Here we only consider $\mathbf{x} = (x_1, ..., x_n)^T$ such that $x_i > 0$ for each $i \in \{1, ..., n\}$, since the X_i are positive almost surely.

The likelihood ratio is given by

$$\begin{aligned}\frac{f_{\mathbf{x}}(\mathbf{x} \mid \lambda_1)}{f_{\mathbf{x}}(\mathbf{x} \mid \lambda_0)} &= \frac{\prod_{i=1}^n \lambda_1 e^{-\lambda_1 x_i}}{\prod_{i=1}^n \lambda_0 e^{-\lambda_0 x_i}} \\ &= \left(\frac{\lambda_1}{\lambda_0}\right)^n e^{-\lambda_1 \sum_{i=1}^n x_i + \lambda_0 \sum_{i=1}^n x_i} \\ &= 2^n e^{-\sum_{i=1}^n x_i}.\end{aligned}$$

Note that we can simplify the inequalities involving the likelihood ratio:

$$\begin{aligned}&\frac{f_{\mathbf{x}}(\mathbf{x} \mid \lambda_1)}{f_{\mathbf{x}}(\mathbf{x} \mid \lambda_0)} > k \\ \Leftrightarrow\ & 2^n \frac{e^{-2\sum_{i=1}^n x_i}}{e^{-\sum_{i=1}^n x_i}} > k \\ \Leftrightarrow\ & g(T(x_1, ..., x_n)) > k \\ \Leftrightarrow\ & T(x_1, ..., x_n) < t = g^{-1}(k)\end{aligned}$$

where $T(x_1, ..., x_n) = \sum_{i=1}^n x_i$, $g(s) = 2^n \exp(-s)$ and so $t = -\log(k) + n\log(2)$. The equivalence holds since g is strictly decreasing.

Under $H_0 : \lambda = \lambda_0 = 1$, $\sum_{i=1}^n X_i \sim \mathrm{G}(n, 1)$ (by independence) has a continuous distribution, therefore the case $\frac{f_{\mathbf{x}}(\mathbf{x}|\lambda_1)}{f_{\mathbf{x}}(\mathbf{x}|\lambda_0)} = k_\alpha$ (which is equivalent to $\sum_{i=1}^n x_i = t_\alpha$) has probability 0, and in particular the middle branch of the NP test does not affect whether $E_{\lambda_0}[d_{NP}(\mathbf{X})] = \alpha$. Therefore, we can arbitrarily choose $\gamma_\alpha = 0$.

The NP test can then be equivalently given by:

$$d_{NP}(\mathbf{x}) = \begin{cases} 1, & \sum_{i=1}^n x_i < t_\alpha \\ 0, & \sum_{i=1}^n x_i \geq t_\alpha. \end{cases}$$

We still need to enforce the condition $E_{\lambda_0}[d_{NP}(\mathbf{X})] = \alpha$ by choosing a suitable value of α. This is equivalent to:

$$\begin{aligned}&P_{\lambda_0}\left(\sum_{i=1}^n X_i < t_\alpha\right) = \alpha \\ \Leftrightarrow\ & P_{\lambda_0}\left(\sum_{i=1}^n X_i \leq t_\alpha\right) = \alpha.\end{aligned}$$

Since $\sum_{i=1}^n X_i \sim \mathrm{G}(n, 1)$ under H_0, this means that $t_\alpha = F_n^{-1}(\alpha)$, for F_n the cdf of the $\mathrm{G}(n, 1)$ distribution.

(b) By definition of the power, we have

$$\beta = E_{\lambda_1}[d_{NP}(\mathbf{X})] = P_{\lambda_1}\left(\sum_{i=1}^n X_i \leq F_n^{-1}(\alpha)\right).$$

Recall that if $Y \sim \mathrm{Exp}(\lambda)$, then $\lambda Y \sim \mathrm{Exp}(1)$. Thus, under $H_1 : \lambda = \lambda_1 = 2$, $2X_1, ..., 2X_n$ are i.i.d $\sim \mathrm{Exp}(1)$, and therefore, by independence, $\sum_{i=1}^n 2X_i \sim \mathrm{G}(n, 1)$. It follows that

$$\beta = P_{\lambda_1}\left(2\sum_{i=1}^{n} X_i \le 2F_n^{-1}(\alpha)\right) = F_n(2F_n^{-1}(\alpha)).$$

(c) We compute $\sum_{i=1}^{10} x_i = 5.367$.

- For $\alpha = 0.05$, $F_{10}^{-1}(\alpha) = F_{10}^{-1}(0.05) \approx 5.425 > \sum_{i=1}^{10} x_i$. Therefore, we reject H_0 with a level of 5%.
- For $\alpha = 0.01$, $F_{10}^{-1}(0.01) \approx 4.130$. Therefore, we cannot reject H_0 with a level of 1% - these data do not present a compelling enough evidence against the null hypothesis.

Exercise 13.3 Again in the setup of exercise 2, it turns out that the Neyman-Pearson test you found in (a) is actually UMP of level α for testing $H_0 : \lambda = 1$ versus $H_1' : \lambda > 1$. More concretely, the same NP test is the most powerful among all tests of level α, for any $\lambda \in \Theta_1' = (1, +\infty)$, and not only for $\lambda \in \Theta_1 = \{2\}$.

Do you see why this is true?

Solution 13.3 Return to the explicit form of the NP-test for this problem:

$$d_{NP}(\mathbf{x}) = \begin{cases} 1, & \sum_{i=1}^{n} x_i < F_n^{-1}(\alpha) \\ 0, & \sum_{i=1}^{n} x_i \ge F_n^{-1}(\alpha). \end{cases}$$

We know (as shown in the lectures) that d_{NP} is a UMP test of level α for testing $H_0 : \lambda = 1$ versus $H_1 : \lambda = 2$. In other words, for any other test d^* such that $E_{\lambda_0}[d^*(\mathbf{X})] \le \alpha$, we would have a lower power:

$$E_{\lambda_1}[d^*(\mathbf{X})] \le E_{\lambda_1}[d_{NP}(\mathbf{X})].$$

However, d_{NP} does not depend on the particular value of $\lambda_1 = 2$. More specifically, if we had to test $H_0 : \lambda = 1$ versus $H_1' : \lambda = \lambda_1'$ for some $\lambda_1' > 1$, we would obtain exactly the same test as above. Since this same test is again UMP of level α, this implies that it is actually UMP of level alpha for the testing problem $H_0 : \lambda = 1$ versus $H_1' : \lambda > 1$.